IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. VII, N. 5 | Serie decima | 1° Marzo 1958

« Composite » Model for K-Mesons.

S. N. BISWAS
Department of Mathematical Physics,
University of Adelaide, South Australia

(ricevuto il 17 Marzo 1957)

Summary. — It is here proposed that the heavy mesons are composite and are excited bound states of two pions. The interaction between the two pions is assumed to be due to a pseudoscalar field with direct coupling. To predict the K-mesons the interaction strength, g^2, is found to be equal to, or rather greater than $1.789\,\pi(\mu/m)$. The usual procedure of the renormalization of the pseudoscalar and scalar theories has been used to obtain the covariant wave equation describing the two-Boson bound system. The complete and separable solutions of the wave equation predict the existence of a new « quantum number » whose properties are similar to those of the « dionic » number or the « strangeness » number proposed by other authors. The decay of the K-mesons has been studied on the basis of a Universal Fermi interaction. The life-time turns out to be $\approx 10^{-7}$ s, in reasonable agreement with the observed time. The associated production of the strange particles has been explained with the help of Feynman diagrams on the assumption of conservation of the new relativistic quantum number.

1. – Introduction.

Several views have been put forward by different authors, in particular, PAIS (1), GELL-MANN (2) and GOLDHABER (3), for the copious production and stability of the strange particles observed in cosmic-ray events. All the theories,

(1) A. PAIS: *Proc. Glasgow Conf.* (1954) (London, 1955), p. 342; *Proc. Nat. Acad. Sci., U.S.*, **40**, 484, 835 (1954); *Proc. Rochester Conf.* (1955).
(2) M. GELL-MANN: *Phys. Rev.*, **92**, 833 (1953).
(3) M. GOLDHABER: *Phys. Rev.*, **92**, 1279 (1953); **101**, 433 (1956).

in fact, stress the need of a new « quantum number » obeying some selection rules, which must be attributed to each of the newly discovered particles. The present experimental discoveries have added to our knowledge of the elementary particles. The particles which are so far known fall into two groups denoted by Y and K groups, commonly called Hyperons and Heavy mesons respectively. Hyperons are unstable particles intermediate in mass between neutron and deuteron and K-mesons are unstable particles intermediate in mass between pion and nucleon. Actually all known particles of the K-group have masses very near to one-half of the nucleon mass. Greek capitals are used for hyperons such Λ, Σ and Ω etc., while small Greek letters for K-particles, e.g. τ, θ etc. When a complete identification has not been possible, the particle is denoted by its group name with subscripts describing decay products. All known new particles have life-times between 10^{-10} s to 10^{-8} s. It is observed that the new particles are produced copiously and their masses are all the same within experimental error. We give below in a tabular form an account of the so-far known properties of the K-meson family.

TABLE I. – *Properties of the* K-*meson family.*

Odd intrinsic parity				Even intrinsic parity			
Particle	Decay-scheme	Frequency	Life-time	Particle	Decay-scheme	Frequency	Life-time
$\theta^\pm(K^\pm_{\mu 2})$	$\mu^\pm + \nu$	58 %	10^{-8}	$\theta^\pm(K_{2\pi})$	$\pi^\pm + \pi^0$	29 %	10^{-8}
$\theta^\pm(\tau'/\tau)$	$\pi^\pm + \begin{cases} 2\pi^0 \\ \pi^\pm + \pi^- \end{cases}$	8 %	10^{-8}	$\theta^\pm \begin{pmatrix} K^\pm_{\mu 3} \\ K^\pm_{e3} \end{pmatrix}$	$\begin{matrix} \mu^\pm \\ e^\pm \end{matrix} + \nu + \pi^0$	6 %	10^{-8}
θ^0	2γ ?	?	10^{-10}	θ^0	$\begin{matrix} 2\pi^0 \\ \pi^+ + \pi^- \end{matrix}$	?	10^{-8}
θ^0	$\pi^0 + \begin{cases} 2\pi^0 \\ \pi^+ + \pi^- \end{cases}$	?	?	θ^0	$\begin{matrix} \mu^\pm \\ e^\pm \end{matrix} + \nu + \pi^\pm$	?	10^{-8}

In the above table we have put a (?)-mark in some places meaning thereby that experimental confirmation has not yet been reached. As for the decay of θ^0 to 2γ-rays there is one confirmation recently, although it has not been generally accepted.

It is evidently desirable to search for some unifying principles which may reveal the properties of these new particles. An attempt of this type has been made in this paper, to find out a suitable model which may explain some of

the properties of the heavy mesons (K-family). Such a model has, however, not been applied here for the hyperons. But it may be remarked that such a theory for K-mesons is not inconsistent with the theories of the other authors. We do not, in detail, outline the theories of the strange particles proposed by various authors, but it may be worthwhile to point out that PAIS ([1]) and GELL-MANN ([2]) hold the view that K-mesons and hyperons are a new fundamental family of particles to be added to the previously established nucleons and pions. GOLDHABER ([3]), however, suggests that K-mesons are a fundamental family of particles while the hyperons are compounds of K-mesons and nucleons. It is supposed that in the observed associated production there is always a pair of θ-mesons one of which might combine with the nucleon to form a hyperon. According to his theory the θ-family is characterized by a new quantum number, d, (called the dionic quantum number) similar to that of the strangeness quantum number « S » of GELL-MANN. In Gell-Mann's scheme it is supposed that the strong interaction responsible for the production of the strange particles, is invariant with respect to arbitrary rotation in the isotopic spin space I_1 and also invariant in another hypothetical charge-space I_2 but only with respect to rotation about the Z-axis; accordingly I_1, I_{1Z} and I_{2Z} are good quantum numbers. His strangeness number is directly connected with I_{2Z} by $S=2I_{2Z}$. PAIS, instead of considering two 3-dimensional rotations, proposed one 4-dimensional rotation about a plane. Gell-Mann's I_1 and I_2 are identical in Pais' theory with the six-components of the angular momentum tensor in an abstract 4-Euclidean space. We, however, would like to hold the view that θ-meson is a composite-particle and accordingly suggest that it is an *excited bound state* of two pions. The interaction forming such a state is a nuclear interaction of pseudoscalar mesons in direct coupling with the nucleons. That a θ-meson is a boson follows of course from the decay schemes $\theta^0 \to \pi^+ + \pi^-$ or $\theta^+ \to \pi^0 + \pi^+$ and from spin conservation.

Accepting the usual procedure of the renormalization of PS(PS) theory we will show that the interaction of the two π-mesons is, at short distances, of the same type as that of the electromagnetic interaction. To this end we have taken the simplest possible Feynman diagrams (see Sect. 2) as our composite-model. It has been found that the wave function describing the excited bound state of two pions must be antisymmetric in isotopic spin indices and that the relativistic parity of the system must be odd. This fact at once leads to the existence of charge-triplets of θ-meson and that they can be further classified with odd or even intrinsic parity obeying the decay scheme shown in Table I. To discuss the nearly equal rate of decay of all such particles it is assumed that the decay is governed by the Universal Fermi interaction and the life-time turns out to be of the order of 10^{-7} s, while the observed rate is 10^{-8} to 10^{-10} s. Another aspect which arises in our present formalism is the

emergence of a new quantum number [4,5] called « relativistic quantum number » this is a feature of the covariant wave equation for a system of particles, having one new degree of freedom (relative time of the two pions concerned here). One should associate a definite value of this relativistic quantum number to characterize the θ-family. It is evidently closely related to the « strangeness » or « dionic » number proposed by other authors. The conservation of the « relativistic quantum number » under strong interactions plays a part in the observed associated production.

It may, however, be remarked that there are several alternative Feynman diagrams possible for the meson-meson system; for example, one can take any higher order diagrams for this purpose including the insertions of meson and fermion self-energy processes. This generalization is absolutely straightforward but it becomes much too involved for practical computations.

It is also worthwhile to mention that while the K-meson is considered here as a composite particle, one does not exclude the alternative of considering it as a new fundamental particle. The nature of the wave function suggests that one may assume the particle as elementary, having a structural co-ordinate similar to that adopted by YUKAWA [6] for the electron.

2. – Feynman amplitude for two-boson systems.

We assume that the K-meson is a composite particle and it is an excited bound state of two pions. The resulting Bethe-Salpeter [7] equation describing the interaction of two pions can be written in the co-ordinate space as

$$\varphi_{ij}(x_3', x_4') = \sum_{i'j'} g^4 \int K_{iji'j'}(x_3', x_4'; x_1, x_2)\, \varphi_{i'j'}(x_1, x_2)\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \,, \tag{2.1}$$

where $\varphi_{ij}(x_3', x_4')$ denotes the Feynman probability amplitude of finding the two particles at x_3' and x_4'. The suffixes i, j denote the isotopic spin indices of the two mesons concerned, g denotes the coupling constant and the interaction is taken as the usual pseudoscalar meson theory with pseudoscalar coupling. The kernel of the integral equation $K_{iji'j'}$ is given by

$$K_{iji'j'} = K^{(a)}_{iji'j'} + K^{(b)}_{iji'j'} + K^{(c)}_{iji'j'} + K^{(d)}_{iji'j'} \,,$$

([4]) S. N. BISWAS and H. S. GREEN: *Nuclear Physics*, **2**, 177 (1956).
([5]) H. S. GREEN and S. N. BISWAS: *Prog. Theor. Phys.*, **18**, 121 (1957).
([6]) H. YUKAWA: *Phys. Rev.*, **77**, 219 (1950).
([7]) E. E. SALPETER and H. BETHE: *Phys. Rev.*, **84**, 1232 (1951).

where $K^{(a)}$, $K^{(b)}$, $K^{(c)}$ and $K^{(d)}$ are obtained from the following (a), (b), (c) and (d) simplest Feynman diagrams (Fig. 1 and 2) for the interaction of the two mesons.

We have,

$$(2.2)\qquad K^{(a)}_{iji'j'} = \int \operatorname{spur}\left\{\tau_i\gamma_5 S(x_1-x_3)\tau_{i'}\gamma_5 S(x_3-x_4)\tau_{j'}\gamma_5 S(x_4-x_2)\tau_j\gamma_5 S(x_2-x_1)\right\}\cdot \cdot \Delta_F(x_3-x_3')\,\Delta_F(x_4-x_4')\,\mathrm{d}^4x_3\,\mathrm{d}^4x_4\,,$$

and $K^{(b)}$ is obtained from $K^{(a)}$ by interchanging x_3 and x_4 in all the S-functions and also the isotopic spin matrices $\tau_{i'}$ and $\tau_{j'}$; $K^{(c)}$ and $K^{(d)}$ are very similar $K^{(a)}$ to and $K^{(b)}$ respectively but only the order of the isotopic spin matrices needs to be changed. The detailed forms will be considered in the Appendix I.

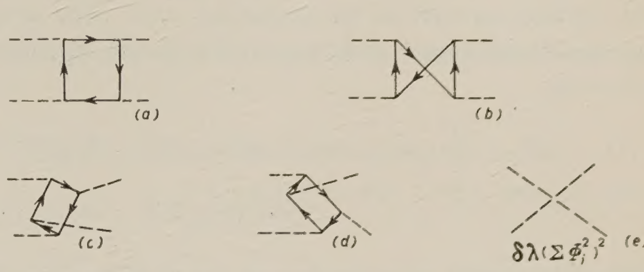

Fig. 1.

Applying the operators $(\Box_3'+\mu^2)$ and $(\Box_4'+\mu^2)$ on both sides of the integral equation (2.1) we have (the indices $3'$ and $4'$ indicate that the differentiations are to be carried out on the respective co-ordinates) the following integro-differential equation,

$$(2.3)\qquad -(\Box_3'+\mu^2)(\Box_4'+\mu^2)\psi_{ij}(x_3', x_4') = = \sum_{i'j'} g^4 \int L_{iji'j'}(x_3', x_4'; x_1, x_2)\,\psi_{i'j'}(x_1, x_2)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,,$$

where now $L_{iji'j'} = L^{(a)}_{iji'j'} + L^{(b)}_{iji'j'} + L^{(c)}_{iji'j'} + L^{(d)}_{iji'j'}$, with

$$(2.4)\qquad L^{(a)}_{iji'j'} = \operatorname{Spur}\left\{\tau_i\gamma_5 S(x_1-x_3)\tau_{i'}\gamma_5 S(x_3-x_4)\cdot \cdot\tau_{j'}\gamma_5 S(x_4-x_2)\tau_j\gamma_5 S(x_2-x_1)\right\},$$

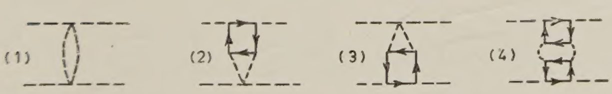

Fig. 2.

and $L^{(b)}$, $L^{(c)}$ and $L^{(d)}$ are obtained similarly from $K^{(b)}$, $K^{(c)}$ and $K^{(d)}$ respectively.

It is clear that the kernel $L_{iji'j'}$ of the integral equation (2.3) is symmetric in its arguments, showing that the wave function ψ_{ij} is also symmetric satisfying Bose-statistics as

required for the K-meson. We have already noted that K-mesons obey Bose-statistics,

Using the center-of-mass frame we can reduce the integro-differential equation (2.3) in momentum space to the following integral equation:

$$(2.5)\qquad -\{(K-l)^2-\mu^2\}\{(K+l)^2-\mu^2\}\psi_{ij}(l)=\sum_{i'j'} g^4\int L_{iji'j'}(K_1 l, l')\psi_{i'j'}(l')\,d^4l'$$

where

$$(2.6)\qquad L_{iji'j'}=L^{(a)}_{iji'j'}+L^{(b)}+L^{(c)}+L^{(d)}$$

(the suffixes as well as the argument have been dropped for convenience in the last three terms) and the explicit forms of them may be seen from the following:

$$(2.7)\qquad L^{(a)}_{iji'j'}=\int \operatorname{spur}\{\gamma_5\tau_i(p_1-m)^{-1}\gamma_5\tau_{i'}(p_1-K-l'-m)^{-1}\cdot$$
$$\cdot\tau_{j'}\gamma_5(p_1-2K-m)^{-1}\gamma_5\tau_{j'}(p_1-K+l-m)^{-1}\}\,d^4p_1\,,$$

where l is relative 4-momentum of the final system. The other terms $L^{(b)}$ etc., can be determined very similarly. In fact if one changes l by $-l$ in the right-hand-side of (2.7) one gets $L^{(b)}$ apart from the necessary changes in the isotopic spin matrices already mentioned. The kernels $L^{(a)}$ etc., as they stand are too formidable to handle and hence in practice suitable approximations must be made. Moreover the integrals of $L^{(a)}$ etc., are strongly divergent, so the usual renormalization procedure of PS(PS) theory should be applied in order to obtain any physically sensible result. For our practical purposes we neglect the meson mass in comparison with that of the nucleon; this neglect ensures the omission of the velocity-dependent terms from the resulting meson-meson potential. The meson-meson potential can be easily determined by taking the Fourier transform of the renormalized approximate kernel. By carrying out a consistent subtraction procedure we will show (see Appendix I) for a special choice of the arbitrary parameter $\delta\lambda_1$ in the renormalization of the pseudoscalar meson theory, that the meson-meson interaction potential has a leading singularity of the type $(s)^{-1}$ with $s=x_\mu x^\mu$ and it is given by

$$(2.8)\qquad \frac{\lambda}{s}\left\{(1+2\gamma)+\log\left(-\frac{1}{4}m^2s\right)+s\log\left(-\frac{1}{4}m^2s\right)+\dots\right\},$$

where m is the nucleon mass, γ, the Euler constant, .577..., and λ is proportional to the coupling constant, the precise value of which will be given later on. If $m^2s\ll 1$, i.e. when the inter-meson distance $< m^{-1}$, m being the nucleon mass, we can replace the expression (2.8) by the leading singularity. By such

an approximation there may arise a source of error but it may be remarked that the nature of the wave function will not materially change if one neglects the logarithmic singularity from the interaction potential. The effect of this approximation may be to some extent to underestimate the value of the coupling constant required to secure binding.

Next we consider the effect of the isotopic spin operators $\tau_i\tau_{i'}\tau_{j'}\tau_j$ etc., appearing in $K^{(a)}_{iji'j'}$ etc., in (2.1). The eigensolutions of such operator equation will designate the possible charge states of the resulting two-pion system. Putting

$$T_1 = \tau_i\tau_{i'}\tau_{j'}\tau_j\,; \qquad T_2 = \tau_i\tau_{j'}\tau_i\tau_{j'}\,; \qquad T_3 = \tau_i\tau_{i'}\tau_j\tau_{j'} \quad \text{and} \quad T_4 = \tau_i\tau_{j'}\tau_j\tau_{i'}$$

we can write (2.6) in the co-ordinate space in terms of the differential equation as follows:

$$(2.9) \qquad -\{(K - i\nabla)^2 - \mu^2\}\{(K + i\nabla)^2 - \mu^2\}\psi_{ij}(x) = \\ = \sum_{i'j'} \lambda f(s)(T_1 + PT_2 + T_3 + PT_4)\psi_{i'j'}(x)\,,$$

where $\lambda f(s)$ stands for the expression given by (2.7) and P is the relativistic Parity operator changing x to $-x$.

Since $\operatorname{spur}\{\tau_i\tau_{i'}\tau_j\tau_{j'}\} = 2(\delta_{ii'}\delta_{jj'} - \delta_{ij}\delta_{i'j'} + \delta_{ij'}\delta_{i'j})$ we can at once find the values of

$$\sum T_1\psi_{i'j'}\,, \qquad \sum T_2\psi_{i'j'}\,, \qquad \sum T_3\psi_{i'j'} \quad \text{and} \quad \sum T_4\psi_{i'j'}\,.$$

It is easily shown that there exists both symmetric and antisymmetric eigen-solutions (see Appendix) with respect to isotopic spin indices. A further examination reveals that the symmetric eigen-solution satisfies a differential equation with a repulsive type of interaction and thus does not allow any bound state; hence only the antisymmetric solution is physically acceptable and this solution satisfies the following differential equation,

$$(2.10) \quad -\{(K - i\nabla)^2 - \mu^2\langle\lambda(K + i\nabla)^2 - \mu^2\}(\psi_{ij} - \psi_{ji}) = 2\lambda f(s)(P - 1)(\psi_{ij} - \psi_{ji})\,,$$

where the relativistic parity is odd, the operator P has the value -1. Thus ψ is odd in the argument, x and hence $\psi(x)\,\delta(x)$ would vanish.

The three possible charge-state solutions are obtained by replacing $\psi_{ij} - \psi_{ji}$ by $\omega_{ij}\psi(x)$ ($\psi(x)$ is now a scalar function) where ω_{12}, ω_{23} and ω_{13} denote the amplitudes of the neutral and the two charge-states of the K-mesons which exist in agreement with experiment.

The equation (2.10) can be further simplified if we neglect the logarithmic singularity from the potential, $f(s)$, exhibiting only the leading singularity,

$(s)^{-1}$, of the potential. The equation reads

$$\{(K - i\nabla)^2 - \mu^2\}\{(K + i\nabla)^2 - \mu\}^2\psi(x) = (\lambda/s)\,\psi(x)\,. \tag{2.11}$$

The equation (2.11) is akin to that considered by WICK (8) and a modification of a method adopted by GREEN (9) can be applied in order to obtain the separable solutions of the equation (2.11). The main point of difference is that WICK predicted only ordinary bound states, while our method will prove the existence of excited bound states. In order that the integral equation represents bound states it is necessary that the transition probability between the bound state and the free states should vanish. If E is half the total energy of the system and P_4 is the relative energy, we have for free particle states $(E + P_4)^2 - \boldsymbol{P}^2 = \mu^2 = (E - P_4)^2 - \boldsymbol{P}^2$ giving $P_4 = 0$ and the wave function (found in the next section) vanishes for $P_4 = 0$. Hence the transition probability vanishes.

A further classification of the K-meson family according to parity and angular momentum considerations will be discussed in a subsequent section. It will be shown that the classification is in agreement with observation. We will also point out that Pais' (10) conjecture—« θ^0 is a particle mixture »—is a direct consequence of such classification arising out of our composite-model theory.

3. – Solution of the wave equation and eigenvalue problem.

Following GREEN (9) we write (2.8) in momentum space as

$$\{(P_4 + E)^2 - \boldsymbol{P}^2 - \mu^2\}\{(P_4 - E)^2 - \boldsymbol{P}^2 - \mu^2\}\varphi(P) = \psi(P)\,, \tag{3.1}$$

where $P \equiv (\boldsymbol{P}, P_4)$ is the relative momentum 4-vector, $K = (\boldsymbol{0}, E)$, E being half the total energy of the system in the center-of-mass frame, and

$$\psi(P) = i\pi^{-2}\lambda\int\{(P - K)^2 - i\varepsilon\}^{-1}\varphi(K)\,\mathrm{d}^4K\,, \tag{3.2}$$

where the $\varepsilon \to +0$ is understood.

Alternatively we have from (3.2) and (3.1)

$$\Box\psi = \frac{4\lambda\psi}{\{P_4 + E)^2 - \boldsymbol{P}^2 - \mu^2\}\{(P_4 - E)^2 - \boldsymbol{P}^2 - \mu^2\}}\,. \tag{3.3}$$

(8) G. C. WICK: *Phys. Rev.*, **96**, 1124 (1954).

(9) H. S. GREEN: *Nuovo Cimento*, **5**, 866 (1957).

(10) M. GELL-MANN and A. PAIS: *Phys. Rev.*, **97**, 1387 (1955).

This differential form of the integral equation (3.2) is equivalent to its integral equation form if it is supplemented by proper boundary conditions (see GOLDSTEIN [11]).

Separable solutions can be obtained by using the following co-ordinate transformation namely

$$(3.4)\qquad \begin{cases} P_1 = P_s \sin\theta\cos\varphi\,, \quad P_2 = P_s\sin\theta\sin\varphi\,, \quad P_3 = P_s\cos\theta \quad \text{and} \\ P_4 + P_s = \pm\, C\coth\left(\dfrac{\xi-\eta}{2}\right) \\ P_4 - P_s = C\,\mathrm{tgh}\left(\dfrac{\xi+\eta}{2}\right) \end{cases}$$

where

$$C^2 = E^2 - \mu^2\,.$$

The whole of the relative energy-momentum space is described by the following different ranges of θ, φ, ξ and η, such that $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant |\xi| \leqslant \infty$ and $0 \leqslant |\eta| \leqslant \infty$. In particular, according to (3.4) we can divide the whole (P_4, P_s)-plane into various regions bounded by the lines

$$|P_4 + P| > C$$

and

$$|P_4 - P| \leqslant C\,.$$

The large values of both P_4 and P_s are obtained by allowing $\xi = \eta = 0$; we have further adopted that the wave function vanishes inside the region bounded by the lines $\xi = \infty$, $\eta = -\infty$; $\xi = -\infty$, $\eta = \infty$; $\xi = -\infty$, $\eta = -\infty$; $\xi = \infty$; $\eta = \infty$; this corresponds to the region covered by the small values of P_4 and P_s: The only part of space considered is that extending from $\xi = \infty$, $\eta = -\infty$ to $\xi = \eta = 0$ and bounded by lines $\xi = \infty$, $\eta = \infty$; $\xi = -\infty$, $\eta = -\infty$ and a similar three others corresponding regions in (P_4, P_s)-plane.

In this new system of co-ordinates we will see that separable solutions of the type

$$(3.5)\qquad \psi = f(\xi)\,g(\eta)\,Y_{l,m}(\theta, \varphi)/P_s$$

exist. Here $Y_{l,m}(\theta, \varphi)$ is as usual the spherical harmonics; $f(\xi)$ and $g(\eta)$ should

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[11] J. S. GOLDSTEIN: *Phys. Rev.*, **84**, 1232 (1951).

satisfy the following differential equations:

$$\frac{d^2g}{d\eta^2} = \left\{n^2 - \frac{l(l+1)}{\cosh^2\eta}\right\} g(\eta) \tag{3.6}$$

and

$$\frac{d^2f}{d\xi^2} = \left\{n^2 - \frac{\lambda}{E^2 + m^2 \sinh^2\xi}\right\} f(\xi)\,, \tag{3.7}$$

where l, m are the orbital and magnetic quantum numbers while n is a different constant. We thus see that a separable solution in the form

$$P_s\psi = f_{E(n)}(\xi)\, g_{n(l)}(\eta) P_l^m(\cos\theta) \exp[im\varphi]$$

exists.

The physical meaning of the $P_l^m(\cos\theta)$ and $\exp[im\varphi]$ are well known; $f_{E(n)}(\xi)$ determines the energy eigenvalue problem being supplemented by proper boundary conditions but the consideration of the problem in a covariant way brings in a new factor $g_{n(l)}(\eta)$. The precise meaning of this new factor has already been pointed out by GREEN [9] and it has no analogue in the non-covariant quantum mechanics. This new constant, n, plays the part of a new quantum number and we will see how it does help in enumerating the classification of the K-meson family.

To find out the required boundary conditions to be satisfied by $f(\xi)$ and $g(\eta)$ we substitute the differential equation for ψ into the original integral equation (3.2). The integral equation is satisfied provided the integrated parts i.e.

$$\frac{\partial f}{\partial \xi}\log\Lambda - f\frac{\partial}{\partial\xi}\log\Lambda \qquad \text{and} \qquad \frac{\partial g}{\partial\eta}\log\Lambda - g\frac{\partial}{\partial\eta}\log\Lambda\,,$$

vanishes at the limits $-\infty$ and ∞, where Λ is a certain well behaved function of ξ and η. This suggests the required boundary conditions are that $f(\xi)$, an even function, and $f'(\xi)$ both vanish exponentially and also $g(\eta)$ and $g'(\eta)$ vanish at $\eta = \infty$. (The prime denotes differentiation).

The general solution of the equation (3.6) for $g(\eta)$ is easily obtained as

$$g(\eta) = \text{constant} \times P_l^n(\text{tgh}\,\eta)\,, \tag{3.8}$$

where $P_l^n(\text{tgh}\,\eta)$ are associated Legendre polynomials in the argument, tgh η. Since $g(\eta)$ has to vanish strongly at $\eta = \pm\infty$ so as to satisfy the boundary conditions we restrict « n » to be an integer. Hence we see that the quantum number « n » is quantized; further selection rules on « n » will be obtained when the wave function is normalized.

In order to solve the equation (3.7) for $f(\xi)$ we notice that the differential equation, (3.7), is Heun's differential equation. The solution satisfying the boundary conditions can be obtained and the convergence criteria for the solution determine a certain transcendental equation for the energy eigenvalue problem. In practice, approximation must be made to solve such a transcendental equation. In view of this purely computational difficulty we would like to make the following approximation:

$$\sinh \xi = 0 \quad \text{for} \quad |\xi| < 1 \qquad \text{and} \qquad \sinh \xi = \infty \quad \text{for} \quad |\xi| > 1\,.$$

The relevant solutions of this differential equation for $f(\xi)$ are given by

$$\text{(3.9)} \qquad \begin{cases} f(\xi) = A \exp\left[-n|\xi|\right] & \text{when } |\xi| > 1\,, \\ = B \cos K\xi & \text{when } |\xi| < 1\,, \end{cases}$$

where $K^2 = ((\lambda/E^2) - n^2)$ and A, B are some constants. The constants can be precisely determined when we normalize the wave-function for the bound system (see Appendix II).

It is evident from the solutions (3.9) that the function $f(\xi)$ satisfies the boundary conditions stipulated above. To determine the energy-eigenvalue we notice that the two solutions of $f(\xi)$, one in the region of the small values of ξ and the other for higher values of ξ must join smoothly. It is obvious that the regularity conditions at $\xi = 1$ give the following energy-eigenvalue equation:

$$\text{(3.10)} \qquad \left(\frac{\lambda}{E^2} - n^2\right)^{\frac{1}{2}} \operatorname{tg} \left\{\left(\frac{\lambda}{E^2} - n^2\right)^{\frac{1}{2}}\right\} = n\,,$$

where n is an integer.

The normalization condition (see Appendix II) also gives that $n > 0$ and $n \leqslant l$. This follows from the normalization integral, $\int_{-\infty}^{\infty} [g(\eta)]^2 \, d\eta$ where $g(\eta) = P_l^n(\operatorname{tgh} \eta)$.

One can solve (3.10) approximately for the energy dependence on λ and n as follows

$$\text{(3.11)} \qquad E^2 = \frac{\lambda}{(\pi/3\alpha)^2 + n^2}\,,$$

where α is some constant, e.g. when $n = 1$, $\alpha \approx 7/6$; $n = 2$, $\alpha \approx 1$ etc. The first excited state corresponds to taking $n = 1$ and $E^2 = \lambda/2$; when $n = 2$, we see $E^2 \approx \lambda/2$, $\lambda/5$; for $n = 3$, $E^2 = \lambda/2$, $\lambda/5$, $\lambda/10$ and so on. The strong interaction for the production of K-mesons suggests that the K-meson corresponds to the value $n = 1$ being the first excited state of the bound two-pion

system. The excitation energy of the K-meson, $E^2 \approx 3.2\mu^2$ (μ is the meson mass) gives the coupling constant $g^2 \approx (1.789(\mu/m)\pi)$. m being the nucleon mass.

It may be pointed out that since the energy-eigenvalue appears quadratically in equation (3.11) one must suggest the existence of anti-particles along with the particles (K-mesons).

4. – Classification of the particles and rate of decay.

We have seen that when $E > \mu$ the acceptable wave function to predict the desired excited bound-state requires the relativistic parity to be odd. Moreover the selection rules for the relativistic quantum number, « n », namely $n \leqslant l$ leads us to assume, for $n = 1$, the values of l, the orbital quantum number, 1. 2, 3, ... etc. Since the K-mesons correspond to $n = 1$ we can have l-values even or odd according as the intrinsic parity is odd or even, so that the whole system may have the relativistic parity that can be determined from the number of pions present in the decay mode. Thus for even intrinsic parity with odd orbital parity ($l = 1, 3, ...$) we have the following mode:

$$\theta^\pm \to \pi^\pm + \pi^0, \qquad \theta^\pm \to \left.\begin{matrix}\mu^\pm \\ e^\pm\end{matrix}\right\} + \nu + \pi^0, \qquad \theta^0 \to \left\{\begin{matrix}2\pi^0 \\ \pi^+ + \pi^-\end{matrix}\right. ; \qquad \theta_4 \to \left.\begin{matrix}\mu^\pm \\ e^\pm\end{matrix}\right\} + \nu + \pi^\mp ;$$

and similarly for odd intrinsic parity with even orbital ($l = 2, 4, ...$) we have

$$\theta^\pm \to \mu^\pm + \nu ; \qquad \theta^\pm \to \pi^\pm + \left\{\begin{matrix}2\pi^0 \\ \pi^+ + \pi^-\end{matrix}\right. ; \qquad \theta^0 \to 2\gamma ; \qquad \theta^0 \to \pi^0 + \left\{\begin{matrix}2\pi^0 \\ \pi^+ + \pi^-\end{matrix}\right. .$$

This classification is in absolute agreement with that shown in the Table I. It is worthwhile to mention that the assumptions of DALITZ [12], and EISENBERG [13] *et al.*, differ from our present scheme. EISENBERG *et al.*, on the assumptions that the interaction radius is of τ-meson Compton wave length and the interaction is independent of the l-values, concluded that the spin and parity of the K-meson is either [0 —], or [1+], or [3+]. Since the spin is the angular momentum in the direction of motion and when the particle is at rest spin is identical with the orbital momentum we see that according to our formalism [1+] and [3+] are allowed, probably a mixture of such states may be the case in reality. Our formalism does not however, support their assumptions. It may be pointed out that Pais' conjecture — θ^0 is a particle mixture—also is consistent with our formalism. According to certain schemes

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([12]) R. H. DALITZ: *Phil. Mag.*, **44**, 1068 (1953).
([13]) Y. EISENBERG, E. LEMON and S. ROSENDORFF: *Nuovo Cimento*, **4**, 610 (1956).

θ^0 possesses an anti-particle $\bar{\theta}^0$ distinct from itself. In field theory they are associated with a complex field ψ say. Denoting the θ^0 of even intrinsic parity with odd orbital parity by the real field ψ_1 and θ^0 of odd intrinsic parity with even orbital parity by another real field ψ_2, we may write the complex field $\psi=(\psi_1+i\psi_2)/\sqrt{2}$ and $\psi^+=(\psi_1-i\psi_2)/\sqrt{2}$. This ψ stands for the θ^0-particle, ψ^+ being the representation for $\bar{\theta}^0$ and corresponding to ψ_1 and ψ_2 we have the quanta θ_1^0 and θ_2^0 respectively. Thus in our theory the parity mixture suggests that θ^0 is a particle mixture as has been shown by PAIS and GELL-MANN [10] from the consideration of the behaviour of the neutral particles under charge-conjugation operators. It follows that their charge conjugation operator is equivalent to our parity operator. Applying the same reasoning to the charged particle states, one sees that a similar degeneracy may arise.

Next, in the following we determine the rate of decay observed in various decay-processes of the θ and τ-mesons. We assume that weak-interactions are responsible for the decay of these unstable particles. We have supposed that the Universal Fermi-interaction governs these observed decay modes. It is well-known that the strength of the Fermi-interaction [14] is of the order of magnitude 10^{-48} to 10^{-49} g cm^5 s^{-2}. We consider here in detail the 3-body decay of K-mesons. The relevant Feynman diagram for the mechanism of the decay mode has been suggested below (Fig. 3). Before we proceed to calculate the probability we note that the life-time of such a decay process may well be found from dimensional considerations, since in the cross-section for the process we have the factor $(F/G)^2$, where F is the universal Fermi interaction constant and G is the dimensionless nuclear coupling constant and the only other thing which comes into play is the meson-mass, μ. To make the life-time a time like quantity the meson mass should be raised to the power μ^5. Thus the life-time turns out $\approx (G/F)^2\mu^{-5}h^7c^{-4}\approx 10^{-7}$ s. A detailed calculation can be given in the following way. The underlying mechanism of the 3-body decay is depicted by the following Feynman diagram (Fig. 3).

Fig. 3. – The θ-meson suffers a Fermi-interaction at one of its vertices shown by the shaded area denoted by F, which gives rise to the emission of e (electron) and ν (neutrino) while the other vertex emits the usual pion.

To determine the lift-time of such a type of decay we need to find out the transition probability of the process shown in the diagram, namely $\theta\to e+\nu+\pi$. We need only calculate the matrix element, M, between the initial state of the θ-meson characterized by « a » and the final state consisting of the decay-products, e, ν, and π, characterized by « b ».

[14] E. FERMI: *Nuclear Physics* (Chicago) p. 73.

The transition amplitude T_{ba} is given by

$$T_{ba} = (2\pi)^4\, \delta^4(\sum P_f - \sum P_i)(b\,|\,M\,|\,a)\,, \tag{4.1}$$

where $\sum P_f =$ total final four-momentum of the system, $\sum P_i =$ total initial four-momentum of the θ-meson and $(b\,|\,M\,|\,a)$ is given by

$$(b\,|\,M\,|\,a) = \left(\frac{F}{G}\right)(2EV)^{-\frac{1}{2}}u(l)(2_{P-l}V)^{-\frac{1}{2}}u_e(P+p)\cdot \cdot u_\nu(l-p)(2VE_{P+p})^{-\frac{1}{2}}(2VE_{l-p})^{-\frac{1}{2}}\,, \tag{4.2}$$

where F is the Universal Fermi constant, G being the nuclear coupling constant, $u(l)$ is the wave function of the θ-meson, $u_e(P+p)$ and $u_\nu(l-p)$ are the spinor functions of the electron and neutrino respectively, V is the normalization volume and E_{P+p} and E_{l-p} are the electron and neutrino final energies respectively.

The decay probability for the process $a \to b$ is celarly given by

$$|T_{ba}|^2 = (2\pi)^8\, \delta^4(0)\, \delta^4(\sum P_f - \sum p_i)\,|(b\,|\,M\,|\,a)|^2\,, \tag{4.3}$$

since $\delta^4(0) = (2\pi)^{-4}VT$, where V is the volume and T is the time during which the transition took place, the transition probability per unit time and volume to a group of final states of momentum P_f is therefore given by

$$\omega_{ba} = (2\pi^4) \sum_{P_f} \delta^4(\sum P_f - \sum P_i)\,|(b\,|\,M\,|\,a)|^2\,. \tag{4.4}$$

To calculate ω_{ba} we use the momentum conservation of the initial and final states and take the square of the expression $(b\,|\,M\,|\,a)$ given in (4.2) and finally integrate over all the final momentum states. In taking the square of the expression $(b\,|\,M\,|\,a)$ we need to consider the summation over all final spin-states of the emitted particles. For the wave function of the θ-meson we use the solution given in (3.5). Replacing the summation $\sum$ in (4.4) by $V^2(2\pi)^{-6}\int d^3p \int d^3l$ we have for (4.4) the following invariant integral over l,

$$\omega_{ba} = \frac{1}{4\pi E}\left(\frac{F}{G}\right)^2 \int d^4l\, u(l)u^*(l)\{(P+l)^2 - m_e^2\}\,\delta_+\{(E-l_4)^2 - l_s - \mu^2\}\,, \tag{4.5}$$

where $u(l)$ is the θ-meson wave function given by (3.5) and $P = (\mathbf{0}, E)$ and $l^2 = l_4^2 - l_s^2$.

The integral (4.5) can be easily evaluated in terms of the new variables, ξ and η, already defined in (3.4). It may be pointed out that since the integral (4.5) is apparently divergent, the usual prescription for the calculation of such

a decay-process, by Fermi interaction, is to cut-off the high momenta at nucleon energy; but in our case we have expressed the transition probability integral in an invariant form, with the advantage that one can adopt, to calculate such an integral, a relativistic cut-off of Feynman. In the integral of (4.5) we introduce the convergence factor,

$$M^4/\{(P+l)^2-M^2\}\{(P-l)^2-M^2\}$$

the limiting procedure $M \to \infty$ after the integration is intended. It is worthwhile to mention that the integral turns out to be absolutely convergent irrespective of the arbitrary cut-off energy.

The lift-time, τ, for the decay is the reciprocal of the transition probability, ω_{ba}, and can be easily found to be

$$\tau = 4\pi\left(\frac{G}{F}\right)^2 \frac{C(E\cosh\xi_0 - C\sinh\xi_0)}{[f(\xi_0)]^2 M_\theta^2 E^3 (E-C)\mu}\,, \tag{4.6}$$

where ξ_0 is given by tgh $\xi_0 = C/E$, E being the excitation energy, $C^2 = E^2 - \mu^2$, μ being the meson mass, i.e. M_θ is half to θ-meson mass and $f(\xi)$ is the solution of equation (3.7) for values of $|\xi| > 1$.

To obtain (4.6) from (4.5) we have used the proper normalization constant of the wave function for the θ-meson and the arbitrary constants A and B associated with the solution of $f(\xi)$ in (3.7) can also be determined from the knowledge of the normalization constant of the θ-meson wave function. The value of the normalization constant has been determined in the Appendix II.

Using the excitation energy $E^2 = 3.2\mu^2$ and $M_\theta = 1.7\mu$ we find the life-time, τ, for the 3-body decay of the θ-meson

$$\tau \approx 1.3\cdot 10^{-7}\ \text{s}\,.$$

Regarding the two-body decay, we assume that the Fermi-interaction instead of taking place at one of the last corner vertices, may happen at an intermediate stage as shown in the Feynman diagram (Fig. 4).

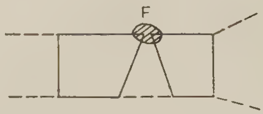

Fig. 4. – F denotes the Fermi-interaction.

In this respect we notice one difference in the decay modes of the neutral and charged particles. In the uncharged particle case, since two neutral mesons cannot be present simultaneously in the intermediate stage, if one of them is to be a neutral pion then the other vertex must suffer Fermi interaction; such a process may happen therefore with greater probability and the life-time may be shorter than in the charged case.

5. – Associated production.

In the following we will consider the mechanism of the associated production of K-mesons by means of appropriate Feynman diagrams Fig. 5. We would only point out that in all observed productions the conservation of the strangeness number or more explicitly the conservation of the covariant angular momentum plays the fundamental role. A similar conservation rule has been applied to the dionic number of GOLDHABER [3]; it may be mentioned that the dionic quantum number of GOLDHABER may have a close similarity to our relativistic quantum number. Hence we can apply a similar argument to those of GOLDHABER in explaining the associated production of the observed strange particles. The conservation of an additional quantum number is also the essence of Pais and Gell-Mann's scheme. We have already discussed their formalism, in brief, in the introduction. It remains to show how our scheme is consistent with that of PAIS. It is well-known that PAIS discussed the properties of the strange particles in terms of a model in which the particles are characterized by quantum numbers corresponding to certain representations of the group of real 4-dimensional rotations in an Euclidean 4-space. Accordingly he introduced the angular momentum tensor

$$K_{ik} = -i\left(x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i}\right) \qquad i, k = 1, \ldots, 4,$$

whose quantization shows the existence of a new constant of motion in addition to those already known in the non-covariant quantum mechanics. In this respect we conclude that PAIS' findings are equivalent to ours. We have considered the covariant relative angular momentum tensor of two particles in center-of-mass frame in the ordinary space-time 4-space instead of considering it in an abstract 4-dimensional Euclidean space as considered by PAIS. The angular momentum operators obey the same commutation rules in both the spaces; their quantization leads, to the existence of a new constant of motion which we have described earlier as « relativistic quantum number ».

The fundamental reaction with which we are concerned is the production of a K-meson and another free $\overline{\text{K}}$-meson (antiparticle) or any hyperons. GOLDHABER assumes that the free $\overline{\text{K}}$-meson may combine with the nucleon present to produce a hyperon. In his model the hyperon is a compound particle of K-meson and nucleon and a K-meson is a fundamental particle, while in our model a K-meson has been described as a composite particle and no definite model has been suggested for the hyperons.

Starting from any combination of a meson and a nucleon we would like to explain the associated production phenomenon on the basis of our compo-

site-model. It is to be shown that the following reactions are allowed, namely

$$\Lambda^0+\theta^0,\quad \Sigma^++\tau^+,\quad K^-+\tau^+,\quad \Sigma^++K^+,\quad \Sigma^-+K^+,\quad \Sigma^++\theta^+,\Lambda^0+K_\mu\quad \text{etc.}$$

while barring at the same time, the possible occurrence of

$$\Lambda^0+\Lambda^0,\quad K^-+K^-,\quad \Sigma^\pm+\Sigma^\pm,\quad \Sigma^-+K^-,\quad \Sigma^++K^-,\quad K^++K^+\quad \text{etc.}$$

In what follows we would confine ourselves to the production phenomenon when one of the particles or both are K-mesons.

a) $\Lambda^0+\Lambda^0$ is not observed: it is impossible to obtain two hyperons from any neutral combination of meson and nucleon. It simply violates the statistics.

b) $\Lambda^0+\theta^0$ is observed. The Feynman diagram Fig. 5 is built to show the production of $\Lambda^0+\theta^0$, starting from an incoming neutral combination of meson and nucleon. In the diagram *b*) we notice that the K^0-meson is produced by the combination of either two π^0's or π^+ and π^-. Of these cases the case 1 is inadmissible because an interchange of the neutral pion lines would yield the same process but since the acceptable wave function is antisymmetric in the spin indices, there will result a destructive interference of the probability amplitudes of the process; thus one is left with the second alternative of the formation of K^0 with π^+ and π^-.

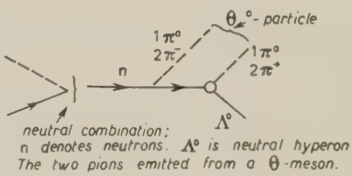

Fig. 5.

c) Σ^++K^+ is observed: The diagram Fig. 6 is drawn from an incoming mixture of π^+ and proton. In Fig. 6 π^+ and π^0 are emitted to form a K^+-meson. In such a case there is no probability of destructive interference of two possible diagrams as obtained in diagram Fig. 5 by the interchange of pion-lines. Hence Σ^++K^+ is observed.

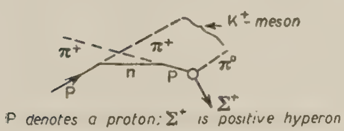

Fig. 6.

d) Σ^-+K^+ is observed: The diagram for this associated production from any neutral combination of meson and nucleon has been drawn in Fig. 7. In this diagram one cannot interchange the pion lines, thus there is no possibility of destructive interference. Hence we get the production of Σ^-+K^+.

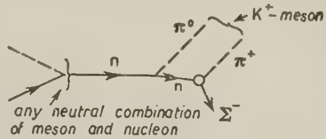

Fig. 7.

e) $\Sigma^+ + K^-$ is not observed: In it we see from the diagram Fig. 8 that one can not observe the process of the production of $\Sigma^+ + K^-$. There are two possible diagrams (both depicted in the same figure by marking 1 and 2) and there is a destructive interference of the probabilities of occurrence due to the anti-symmetrical nature of the wave function of the emitted K^--meson, in the isotopic spin indices. Hence it cannot be detected.

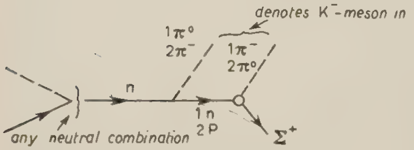

Fig. 8.

f) $K^+ + K^-$ process has been detected. The diagram Fig. 9 may be built up from any incident neutral combination of meson and nucleon. The various possibilities of the intermediate and final stages have been labelled by 1a, 1b, 2a and 2b. If we consider now the different processes separately, we notice that case 1a and 2a may occur with destructive interference simply because the interchange of pion-lines will yield the same process but with different signs of the amplitudes; hence by these two mechanisms $K^+ + K^-$ production is impossible. On the other hand cases 1b and 2b are purely possible, the first case allowing K^+ and K^- and the second giving $K^- + K^+$.

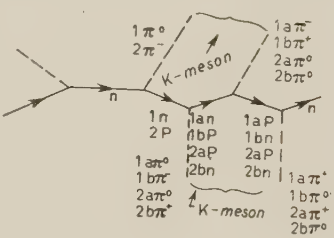

Fig. 9.

g) Production of $K^+ + K^+$ not detected: Possible Feynman diagrams are shown in Fig. 10. The incoming particles are proton and π^+-meson. Here also there are two possibilities denoted by 1 and 2 and according to previous argument such amplitudes occur with destructive interference.

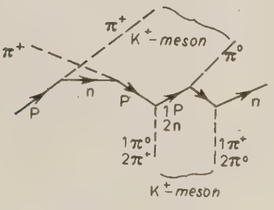

Fig. 10.

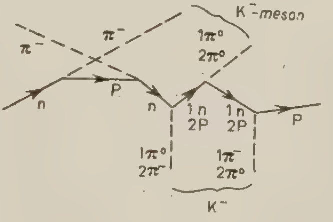

Fig. 11.

h) $K^- + K^-$ also not observed. The argument as in *g*) will show that the production can not be detected. The diagrams Fig. 11 are drawn and various intermediate diagrams are denoted by 1 and 2.

i) $\Lambda^0 + K_\mu$ observed: The diagrams Fig. 12 (1) and (2) are drawn from any incoming mixture of meson-nucleon system to give rise to proton. The diagram (1) cannot be detected while (2) gives the correct description of the production of $\Lambda^0 + K_\mu^+$. The diagram (3) will show the production probability of $\Sigma^0 + K_\mu^-$.

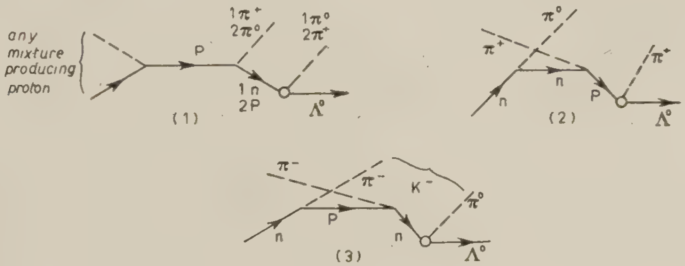

Fig. 12.

In drawing all these Feynman diagrams we have excluded the neutral combination of incoming particles when the incoming meson is absorbed after a pion for the formation of θ or a hyperon being emitted. This gives a violation of the strangeness.

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The author is grateful to Professor H. S. Green, for suggesting the problem and for his constant help and guidance, and finally for his valuable criticism over the manuscript, which led to its material development. The author's sincerest thanks are also due to Dr. C. A. Hurst for kindly going through the manuscript and for his suggestions for its improvement.

Appendix I.

Renormalization of the kernel and effective meson-meson potential.

The integro-differential equation satisfied by the Feynman probability amplitude for the meson-meson interaction is given in a symbolic way

$$\text{I(1)} \qquad \boldsymbol{O}\psi_{ij} = (L^{(a)}_{iji'j'} + L^{(b)}_{iji'j'} + L^{(c)}_{iji'j'} + L^{(d)}_{iji'j'})\psi_{i'j'}\,,$$

where $\boldsymbol{O}$ is the differential operator given in the left-hand-side of equation (2.3) and the summation and integration signs are understood on the right-hand side.

The structure of the $L^{(a)}_{iji'j'}$ etc., is given by

I(2) $$L^{(a)}_{iji'j'} = \operatorname{spur} \tau_i\gamma_5 S(x_1 - x_3)\tau_{i'}\gamma_5 S(x_3 - x_4)\tau_{i'}\gamma_5 S(x_4 - x_2)\tau_i\gamma_5 S(x_2 - x_1)\,.$$

$L^{(b)}$ is obtained from $L^{(a)}$ by interchanging x_3 and x_4 and as well the τ-matrices $\tau_{i'}$ and $\tau_{j'}$. We abbreviate as follows

I(3) $$\begin{cases} L^{(a)}_{iji'j'} = L^{(a)} \operatorname{spur}\{\tau_i\tau_{i'}\tau_{j'}\tau_j\}\,; & L^{(b)}_{iji'j'} = L^{(b)} \operatorname{spur}(\tau_i\tau_j\tau_{i'}\tau_{j'}) \\ L^{(c)}_{iji'j'} = L^{(c)} \operatorname{spur}\{\tau_i\tau_{i'}\tau_j\tau_{j'}\} & L^{(d)}_{iji'j'} = L^{(d)} \operatorname{spur}\{\tau_i\tau_{j'}\tau_j\tau_{i'}\}\,, \end{cases}$$

when the explicit form of $L^{(a)}$ etc., can be obtained as

I(4) $$L^{(a)} = \operatorname{spur}\left\{\gamma_5 S(x_1 - x_3)\gamma_5 S(x_3 - x_4)\gamma_5 S(x_4 - x_2)\gamma_5 S(x_2 - x_1)\right\}.$$

Similarly $L^{(b)}$ is the same as $L^{(a)}$ only x_3 is replaced by x_4. The form of $L^{(c)}$ is identical with that of $L^{(a)}$ and $L^{(d)}$ with that of $L^{(b)}$.

For our purpose we find it convenient to work in the momentum space. The Fourier transform of $L^{(a)}$'s etc., can be easily obtained; we hovever, give a special form of $L^{(a)}$, others easily follow. The form of $L^{(a)}$ is given in Sect. **2**, eq. (2.7). For our practical purpose we neglect the meson-mass in comparison with that of the nucleon; this neglect is equivalent to throwing away the velocity-dependent term from the potential. From Sect. **2** eq. (2.7) we have, changing the variables by the substitution $q_1 = p_1 + l'$ and $q = l - l'$, and performing the spur calculations

I(5) $$L^{(a)}(q) = \int\left\{\frac{q_1^2 - m^2 + q_1\cdot q}{(q_1^2 - m^2)^2\{(q_1 + q)^2 - m^2\}}\right\} \mathrm{d}^4 q_1\,.$$

Similar expressions can be obtained for the other $L^{(b)}$, $L^{(e)}$ and $L^{(d)}$. One readily sees that the integral (Eq. (5)) of $L^{(a)}(q)$ is a divergent one. We now apply a consistent subtraction procedure in order to leave behind an absolutely convergent integral. It is well-known that in PS(PS) and S(S) theories besides the primitive divergences in electrodynamics, a new type of divergence occurs corresponding to the Møller scattering of two mesons (see Fig. 1). This divergence is removed by adding consistently an interaction of the type $\delta\lambda\varphi^4$ ($\delta\lambda$ is an infinite constant; this form is appropriate for neutral PS(PS) theory); since we are considering here symmetrical PS(PS) theory we need to modify this infinite counter-interaction term to $\delta\lambda(\sum_i \varphi_i^2)^2$, $i = 1, 2, 3$, where φ_i's are meson variables.

Subtracting the divergence part from the kernel of (1) we write the renormalized equation (denoted by superfix « r »)

I(6) $$O\psi^r_{ij} = L^{(r)}_{iji'j'}\psi^{(r)}_{i'j'}\,,$$

where the explicit form of $L^{(r)}_{iji'j'}$ is

I(7) $$L^{(a)}_{iji'j'} - A^{(a)}_r\delta_{ii'}\delta_{jj'} + (L^{(b)}_{iji'j'} - A^{(b)}_r\delta_{ii'}\delta_{jj'}) + \\ + (L^{(c)}_{iji'j'} - A^{(c)}_r\delta_{ii'}\delta_{jj'}) + L^{(d)}_{iji'j'} - A^{(d)}_r\delta_{ii'}\delta_{jj'})\,.$$

The sum of the divergences namely $(A_r^{(a)}+A_r^{(b)}+A_r^{(c)}+A_r^{(d)})$ must be cancelled from the counter term, i.e. from the expectation value of $\langle\varphi_k\varphi_k\varphi_l\varphi_l\rangle$ obtained from the S-matrix element due to the additional interaction $\delta\lambda(\sum\varphi_i^2)^2$. (See Fig. 1 ($e$)).

In practice we demonstrate the idea by carrying out the indicated procedure on one of the $L^{(a)}$'s. In general the method [15] is straightforward. We accordingly subtract from the integrand (5) the expression obtained by putting $q=0$ in the original integrand itself. This procedure leaves behind for the $L^{(a)}(q)$, the following

I(8)
$$L^{(a)'}(q) = -\int \frac{q(q+q_1)\,\mathrm{d}^4q_1}{(q_1^2-m^2)^2\,\{(q_1+q)^2-m^2\}}\,.$$

The direct Fourier transform of $L^{(a)'}(q)$ should give us the effective meson-meson potential. But a further examination of the integral $L^{(a)'}(q)$ shows that $L^{(a)'}(q)$ does not tend to zero when $q\to\infty$ and hence the Fourier transform will be singular at the origin. Near the origin the most singular part is $\delta(x)(\partial/\partial m^2)\Delta(s)+(1/4\pi^2s^2)+(2.8)$. We neglect the $\delta(x)$ part because the wave function vanishes at the origin.

We may notice that on account of the inclusion of $\delta\lambda(\sum\varphi_i^2)^2$ in the Lagrangian, there are other possible diagrams to be considered (see diagrams Fig. 2). The diagrams (2), (3), (4) of Fig. 2 are required for their divergent contributions so as to cancel the divergences appearing from the second-order direct meson-meson scattering (diagram (1) of Fig. 2). It has been pointed out by WARD [16] that the construction of finite functions which replace the infinite functions in the calculation of the renormalized S-matrix, is indeterminate by an arbitrary finite constant ($\delta\lambda_1$). The choice of this empirical constant can be effected in the following way. We have mentioned that the integral (8) defining the renormalized kernel behaves near the origin in co-ordinate-space as a strongly singular function like s^{-2}. The second-order matrix element for the direct meson-meson scattering can be calculated in an usual way with the interaction Hamiltonian $\delta\lambda(\sum\varphi_i^2)^2$, where $\delta\lambda$ contains a finite part $\delta\lambda_1$ [16]. It is easily found that the above scattering-matrix contributes a term which in the co-ordinate space is equal to $-(\delta\lambda_1)^2s^{-2}$. The corresponding $(s)^{-2}$ singularity in the kernel (8) is $+(g^4/4\pi^2)s^{-2}$. Hence by choosing $(\delta\lambda_1)^2$ to be equal to $g^4/4\pi^2$ we can cancel this highly singular part from the interaction kernel just leaving behind a leading singularity of $(s)^{-1}$ in the resulting meson-meson potential. The result of this consistent subtraction procedure enables us to get the final meson-meson potential in the form given by Eq. (2.8), Sect. **2**. Any other choice of $\delta\lambda_1$ would not give bound states with finite energies. Whether this is a correct choice of $\delta\lambda_1$ could be ascertained from scattering experiment. We may note that $L^{(b)}(q)$ is obtained from $L^{(a)}(q)$ by changing the sign of momenta l. Equivalent procedure will be to change the sign of x (change x by $-x$). But eq. (2.8), Sect. **2**, remains invariant under such a transformation. Similar treatment occurs for $L^{(c)}$ and $L^{(d)}$.

As regards the charge-triplet solutions we may note the following. Putting

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(15) A. SALAM: *Phys. Rev.*, **82**, 217 (1951).
(16) J. C. WARD: *Phys. Rev.*, **84**, 897 (1951).

$T_1 = \tau_i \tau_{i'} \tau_{j'} \tau_j$, $T_2 = \tau_i \tau_{j'} \tau_{i'} \tau_j$, $T_3 = \tau_i \tau_{i'} \tau_j \tau_{j'}$ and $T_4 = \tau_i \tau_{j'} \tau_{j'} \tau_i$ we see that

$$\sum T_1 \psi_{i'j'} = 2(\psi_{ij} - \psi_{ji} + (\mathrm{Sp}\,\psi)\delta_{ij})\,; \quad \sum T_2 \psi_{i'j'} = 2(\psi_{ji} - \psi_{ij} + (\mathrm{Sp}\,\psi)\delta_{ij})\,,$$
$$\sum T_3 \psi_{i'j'} = 2(\psi_{ij} + \psi_{ji} - (\mathrm{Sp}\,\psi)\delta_{ij})\,; \quad T_4 \psi_{i'i'} = 2(\psi_{ji} + \psi_{ij} - (\mathrm{Sp}\,\psi)\delta_{ij})\,.$$

From this we easily verify that the symmetric solution obeys

I(9) $$O(\psi_{ij} + \psi_{ji}) = 2\lambda f(s)(1 + P)(\psi_{ij} + \psi_{ji})$$

and the anti-symmetric solution obeys

I(10) $$O(\psi_{ij} - \psi_{ji}) = 2\lambda f(s)(P - 1)(\psi_{ij} - \psi_{ji})\,,$$

where P is the relativistic parity operator; (9) cannot predict any bound state while (10) can if we restrict ourselves to the odd value of P.

Appendix II.

Normalization of the wave function.

Here we would find out the normalization condition of the wave function describing the excited bound state of two-pion system which as mentioned by various authors ([17]), will serve to distinguish the physically significant solutions from the entire continuum of solutions. The normalization condition will directly lead us to determine the normalization constants associated with the wave function. The method adopted here is closely analogous to that used by Allcock ([18]) for the bound state of two-fermions although the derivation is slightly different from this treatment.

A large amount of information concerning the bound-state system is obtained from the appropriate Feynman amplitude which obeys the integral equation of the type

II(1) $$\psi(x, x') = \psi_0(x, x') + \int \psi_0(x, x') K(x_1) \psi(x_1, x')\, \mathrm{d}^4 x_1\,,$$

where ψ_0 satisfies $-(\Box_x^2 + \mu^2)(\Box_{x'}^2 + \mu^2)\psi_0 = \delta(x - x')$; K determines the interaction of the system.

We abbreviate II(1) by the following and may consider it in the momentum space

II(2) $$\psi = \psi_0 + \psi_0 K \psi\,.$$

([17]) S. Mandelstam: *Proc. Roy. Soc.*, A **233**, 248 (1955).

([18]) G. R. Allcock: *Phys. Rev.*, **104**, 1799 (1956).

If there exists a mass state M ($\frac{1}{2}$ mass of the two pion system), we may write $\psi = \varphi/(p^2 - M^2)$ and II(2) reduces

$$\text{II(3)} \qquad \varphi(p) = (p^2 - M^2)\psi_0 + \psi_0 K\varphi .$$

The meaning of the operator equation II(3) is clearly that the 2nd term on the right-hand side is supplementary by proper integration signs. Now to obtain the normalization condition we follow the method used in proving the Ward identity [19] of the renormalized field theory. Consider the free-particle system satisfying $P^2 = M^2$. Then we have from II(3)

$$\text{II(4)} \qquad \varphi(P) = \psi_0 K\varphi(P) .$$

Differentiation II(3) on both sides with respect to p we have

$$\frac{\partial}{\partial p}\varphi = 2p\psi_0 + (p^2 - M^2)\frac{\partial}{\partial p}\psi_0 + \frac{\partial\psi_0}{\partial p}K\varphi + \psi_0\frac{\partial K}{\partial p}\varphi + \psi_0 K\frac{\partial}{\partial p}\varphi .$$

Introducing an arbitrary function $\theta(P)$ which satisfies the same equation II(4) as $\varphi(P)$ we have for $p = P$

$$\text{II(5)} \qquad O = 2M^2\theta + \theta L\varphi ,$$

where L is given by

$$\text{II(6)} \qquad L = \lambda K P\cdot\frac{\partial\psi_0}{\partial P}K + P\cdot\frac{\partial K}{\partial P} .$$

Writing

$$\varphi(x, x') = u(x)v(x') ,$$

where $u(x)$ and $v(x')$ satisfy the same homogeneous integral equation as φ and they are properly normalized, we see that II(5) reduces to the following in the co-ordinate space,

$$\text{II(7)} \qquad O = 2M^2 + \iint L(x_1, x_2)\varphi(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 ,$$

where $L(x_1x_2)$ is the fourier transform of II(6).

The equation II(7) is the desired condition for normalizability.

For our purpose we determine the normalization constant by using II(7) in momentum space and write

$$\text{II(8)} \qquad O = 2M^2 + \iint u(l)L(l, l_1)v(l_1)\,\mathrm{d}^4l\,\mathrm{d}^4l_1 .$$

([19]) J. C. Ward: *Phys. Rev.*, **77**, 293 (1950).

where $u(l)$, $v(l)$ and $L(l, l_1)$ are the Fourier transforms of $u(x_1)$, $v(x)$ and $L(x, x')$ In evaluating II(8) we have replaced $u(l)$ and $v(l)$ by the function $\varphi(l)$ given by Eq. (3.1), Sect. 3; since both satisfy the same integral equation only that they may be proportional by some constants. For L we use II(6) and ψ_0 is obtained from the momentum representation of II(1) and given by

$$\psi_0(l) = \frac{-1}{\{(E+l_4)^2 - \boldsymbol{l}^2 - \mu^2\}\{(E-l_4)^2 - \boldsymbol{l}^2 - \mu^2\}}$$

and K, the interaction function, is given by $(l - l')^{-2}$.

Using the above-mentioned transformation we can at once obtain, for the integral in the right-hand side of II(8), the following

$$\text{II(9)} \quad \int [\psi(l)]^2 \; 2K^2\{K^2 + l^2 - \mu^2\} - 4(K\cdot l)^2]\{(K-l)^2 - \mu^2\}^{-2}\{(K+l)^2 - \mu^2\}^{-2}\,\mathrm{d}^4 l$$

with $K = (\boldsymbol{O}, E)$.

The integral II(9) has been calculated with the transformation given in Sect. 3 eq. (3.4); and $\psi(l) = Nf(\xi)g(\eta)Y_{l.m}(\theta, \varphi)/l_s$, where N is the normalization constant. The substitution of the value of the integral II(9) back into II(8) yields the result:

$$N^2 = 4M^2E^2 ;$$

in this calculation we have tacitly assumed that $f(\xi)$ and $g(\eta)$ are normalized to unity and also $\int [Y_{l,m}(\theta, \varphi)]^2 \mathrm{d}\Omega = 1$. The fact $\int_{-\infty}^{\infty} [g(\eta)]^2 \mathrm{d}\eta = 1$ suggests the selection rules $n \leqslant l$ and $n > 0$ as may be seen from the explicit expression of $g(n)$. Similarly $\int_{0}^{\infty} f^2(\xi)\,\mathrm{d}\xi = 1$ ensures that the integration constants $B^2 = \{\frac{1}{2}(1 + (\sin 2K)/2K)\}^{-1}$ and $A^2 = e^{2n}(1 + \cos 2K)(1 + \sin^2 K)^{-1}$.

RIASSUNTO (*)

Si pone nel presente lavoro che i mesoni pesanti siano composti, siano cioè gli stati legati eccitati di due pioni. Si assume che l'interazione tra i due pioni sia dovuta a un campo pseudoscalare con accoppiamento diretto. Si trova che affinchè possano risultarne mesoni K l'intensità d'interazione, g^2, deve essere uguale a, o meglio maggiore di, 1.789 $\pi(\mu/m)$. Si è seguito il consueto procedimento della rinormalizzazione delle teorie pseudoscalare e scalare per ottenere l'equazione d'onda covariante che descrive il sistema legato di due bosoni. Le soluzioni complete e separabili dell'equazione d'onda predicono l'esistenza di un nuovo « numero quantico » le cui proprietà sono simili a quelle del « numero dionico » o al « numero di stranezza » proposto da altri autori. Il decadimento del mesone K è stato studiato sulla base di una interazione universale di tipo Fermi. La vita media risulta essere $\approx 10^{-7}$ s, in ragionevole accordo col tempo osservato. La produzione associata delle particelle strane si spiega con l'ausilio di diagrammi di Feynman assumendo la conservazione del nuovo numero quantico.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

The Simple Cones of Albedo of Cosmic Rays.

R. Gall

Instituto de Geofísica, Instituto Nacional de la Investigación Científica, México, D. F., México

J. Lifshitz

Instituto Nacional de la Investigación Científica, México, D. F., México

(ricevuto il 6 Agosto 1957)

Summary. — Simple albedo cones for 5 points of incidence were evaluated for different proton energies. The latitude, longitude and energy effects are discussed. Some results are compared with those of Schremp. The altitude, above which the interaction with the atmosphere does not invalidate the results, is given. The validity of albedo model for different altitudes is discussed. The points of emission were studied.

1. – Introduction.

The evaluation of the cones of albedo of cosmic rays was based on the computation of approximately 1000 trajectories of secondary protons in the magnetic field of the earth [1,2]. The trajectories are computed by numerical integration using the method suggested by Milne [3].

Choosing a velocity equal to 1 and taking the earth's radius to be 0.5, the differential equation of motion becomes.

$$\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = C\,\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\left[\nabla\left(\frac{z}{r^3}\right) + \alpha\nabla\left(\frac{xy}{r^5}\right)\right]. \tag{1}$$

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(1) R. Gall and J. Lifshitz: *Proceedings of Cosmic Ray Congress*, Guanajuato, México, September 1955 (in print).

(2) R. Gall and J. Lifshitz: *Phys. Rev.*, **101**, 1821 (1956).

(3) W. E. Milne: *Numerical Calculus* (Princeton, 1949), p. 135.

C is inversely proportional to the magnetic rigidity of the particle: $C = -m_1 aq/4pc$; where a is the earth's radius, equal to $6.37 \cdot 10^8$ cm, q is the charge of the proton in esu, c the velocity of light, p the relativistic momentum of the particle. Also, $\alpha = 3m_2/4m_1$ a constant; here $m_1 = M_d/a^3 = 0.3097$ G, $m_2 = M_c/a^4 = 0.0024$ G, and M_d, M_c are the dipole and quadrupole moments, respectively.

The model used for the magnetic field is that of SCHMIDT and CHARGOY ([4,5]). It consists of a dipole and a quadrupole, both eccentric, the axes of which form an orthogonal system. The origin of this system resides at the point (— 344, 150, 96) in km, given in geographic coordinates ([6]). This geomagnetic system was used as the system of reference.

Five incident points on the earth's surface were chosen (see Table I).

TABLE I. – *Geomagnetic Coordinates for the Points of Incidence.*

Point	1	2	3	4	5
λ	0°	0°	0°	30° N	70° N
φ	0°	45° E	135° E	22.5° E	0°
r	.49581	.52027	.52161	.50686	.49581
H_c/H_d %	10.9	15.6	15.6	8.6	2.0

λ is the geomagnetic latitude, φ the geomagnetic longitude, r the distance from the geomagnetic center, H_d, H_c the dipole and quadrupole fields respectively.

The trajectories were computed for protons of energies .172, .5, 2.0 GeV. These energies fall within the spectrum of secondary protons. The energy of .172 GeV was selected in order to compare the albedo cone with the simple shadow cone for 70° latitude ([7]).

In the numerical integration, the values of the position, geomagnetic field, velocity and acceleration were computed for points along the trajectory, separated by a distance of 30.578 km, for high energies, and by a distance of 15.289 km, for lower energy of protons. At each point, two approximations to the velocity and acceleration were obtained.

2. – Albedo cones.

We shall define the simple albedo cone as the solid angle of directions along which particles of given energy arrive at a point of incidence from other

([4]) S. CHAPMAN and J. BARTELS: *Geomagnetism*, II (Oxford, 1940), p. 651.
([5]) A. CHARGOY: *Revista Mexicana de Física*, **2**, 1 (1953).
([6]) Based on the map of the geomagnetic field of 1945.
([7]) E. J. SCHREMP: *Phys. Rev.*, **54**, 158 (1938).

points of the earth without a single intermediate local minimum. The limiting directions present a local intermediate minimum tangent to the earth's surface.

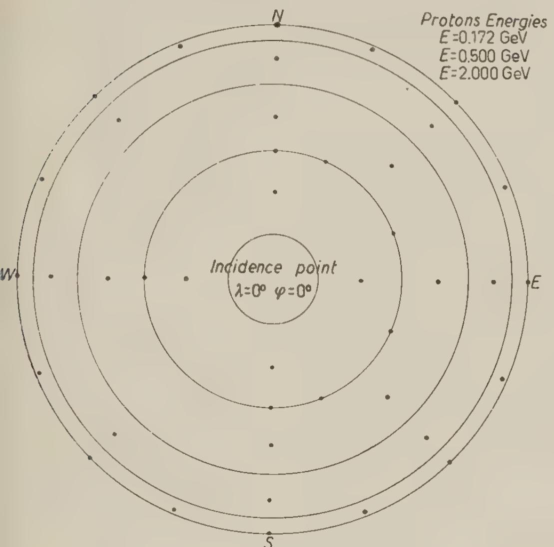

Fig. 1 (*).

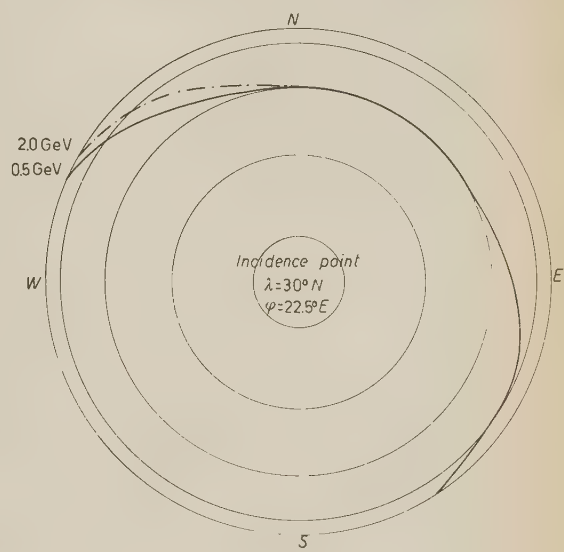

Fig. 2.

We shall also define the higher order albedo cones, the second, the third, etc., as the solid angles of directions along which particles arrive from other points of the earth having one, two, etc., intermediate local minima.

The limiting directions were obtained by interpolation of orbits of the same azimuthal angle.

Fig. 1 shows completely open simple albedo cones for three different energies of protons at the point 1 of incidence (see Table I).

In Fig. 2, two simple albedo cones for 0.5 and 2.0 GeV, at the incidence point 4 are shown. Wi-

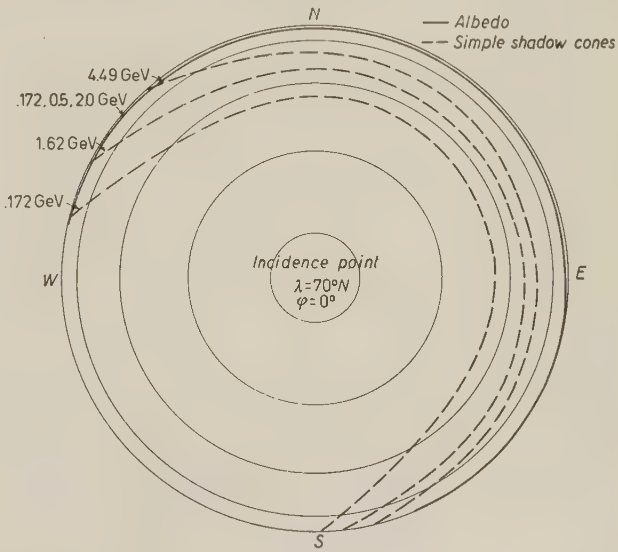

Fig. 3.

(*) In the four figures the Azimuth and Zenith angles are geographic.

thin the cones, the north-eastern directions are predominant. The directions outside the cones must belong to the higher order albedo cones as no direction for these energies can come from infinity. At the point of incidence studied, the smaller the energy the larger the cone.

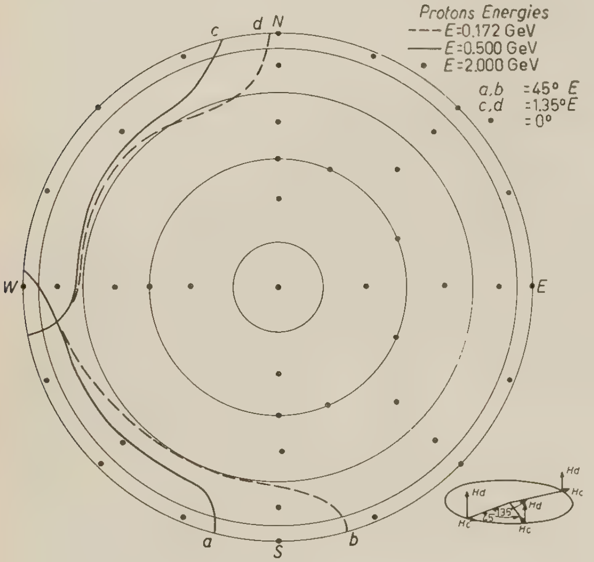

Fig. 4.

For point 5 of incidence, the solid curve (see Fig. 3) represents simple albedo cones for secondary protons of the three energies studied. We can see that the cones are almost completely closed except for some directions of large zenithal angles. The directions outside the cones either belong to the higher order albedo cones or extend to infinity.

The dotted lines represent the simple shadow cones as calculated by SCHREMP (7). A noticeable difference between the simple albedo and the simple shadow cones exists. This difference shall be investigated in our further studies.

The longitude effect is caused both by the eccentricity of the geomagnetic center and by the quadrupole field.

In Fig. 4, both effects can be studied for points at the geomagnetic equator. The little drawing on the right shows the directions of the magnetic field at three points. The dipole is largest at point 1, as this point is closest to the geomagnetic center. The relative magnitudes of the quadrupole to the dipole for these three points are given in Table I. As points 2 and 3 are equidistant from the magnetic center, the differences of their cones are due to the quadrupole field effect. The eccentricity effect can be observed in comparing the cones of point 1 with the cones of either point 2 or 3. For point 1 the cones are completely open while for point 2 (curves a, b) or for point 3 (curves c, d) some directions are closed. The eccentricity has the effect of opening the cone as the point of incidence is situated nearer to the geomagnetic center. The quadrupole effect shifts the cones from north-western to south-western directions.

3. – Points of emission.

The points of emission, which send the orbits towards the points of incidence along the directions within the simple albedo cones, were investigated. From the study of their geomagnetic co-ordinates, it was found that for points 4 and 5 of incidence, the latitudes and longitudes of the points of emission fall within a band not greater than $\pm 5°$ from the point of incidence. For the incidence points at the equator, the emission points are situated mostly to the east of the incidence 'point and fall within a band of $\pm 14°$ latitude and $6°$ longitude.

4. – Albedo cones for different altitude observers.

The albedo cones for the proton energies of .172, .5 and 2.0 GeV calculated for an observer situated at the earth's surface are also valid for an observer situated at h km above the earth's surface, for energies slightly smaller than the above mentioned.

For an observer situated at h km, the differential equation of motion becomes

$$\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = C_h \frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\left[\nabla\left(\frac{z}{r^3}\right) + \alpha_h \nabla\left(\frac{xy}{r^5}\right)\right], \tag{2}$$

where

$$C_h = -\frac{m_1 a q}{4}\,\frac{1}{(cp)(1+h/a)^2}, \qquad \alpha_h = \alpha(1+h/a)^{-1}.$$

TABLE II. – *Data for Different Altitudes' Observers.*

h km	x g/cm²	h/a	$(cp)_h$ GeV	E_h GeV	Points — d_h % 1	2	3	4	5
0	1033.3	0	0.594 1.090 2.784	0.172 0.500 2.000	0	0	0	0	0
107.00	$3\cdot10^{-4}$	0.017	0.574 1.054 2.693	0.161 0.473 1.913	0.18	0.25	0.25	0.14	0.03
318.56	$2\cdot10^{-8}$	0.050	0.538 0.989 2.525	0.144 0.425 1.756	0.51	0.73	0.73	0.39	0.09

Here h is the altitude of the observer; x, the atmospheric depth; E_h are the energies corresponding to the same cones for different altitude observers; H_h, H are the fields in equations 1 and 2, respectively, where the units of length are diffierent; d_h% gives the relative difference between these two fields.

The unit length is chosen to be $2(a+h)$ km where a is the earth's radius; $(cp)_h$ is expressed in terms of this unit of length. Choosing the energy such that $(cp)_h = (cp)(1+h)/a)^{-2}$ we obtain $C_h = C$.

With these values, the equation for an « h » observer differs very slightly from the equation for the earth's observer. The difference resides in the quadrupole field term and is not greater than 0.73% for an altitude as high as 319 km. The difference does not affect the albedo cones.

It is to be understood that both the points of emission and the points of incidence are situated at the altitude of h km.

5. – Interaction with the atmosphere.

The albedo trajectories have been evaluated without taking into account the interaction of the secondary particles with the atmosphere. However, the albedo model describes with sufficient approximation the trajectories of secondary protons interacting with the atmosphere, provided both the points of incidence and emission are situated at an altitude of 107 km or higher. This result is based on the comparison between the imprecision due to the numerical integration and the magnitude of the perturbation due to the interactions with the atmosphere.

The discussion of the validity of the albedo model for describing the trajectories of protons moving in the atmosphere must however be post-poned to the time when the magnetic field in the earth's atmosphere will be better known.

* * *

The authors express their thanks to Dr. MANUEL SANDOVAL VALLARTA under whose direction this work has been performed. We would also like to thank Mrs. MANUELA ALVAREZ, Mrs. LUCILA CAMACHO and Mrs. GRACIELA OYARZABAL for their most valuable help in the computation of the trajectories, as well as Mr. ALBERTO GARCÍA for his excellent collaboration in the operation of the IBM calculating machines.

RIASSUNTO (*)

Per 5 punti d'incidenza sono stati calcolati i coni di albedo semplice per differenti energie protoniche. Si discutono gli effetti di latitudine, longitudine ed energia. Alcuni dei risultati si confrontano con quelli di Schremp. Si dà l'altezza oltre la quale l'interazione con l'atmosfera non infirma i risultati. Si discute la validità del modello d'albedo per differenti altezze. Si sono studiati i punti di emissione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

On the Validity of Levinson's Theorem for Non-Local Interactions (*).

A. Martin

Laboratoire de Physique, École Normale Supérieure, Université de Paris

(ricevuto il 4 Ottobre 1957)

Summary. — By introduction of an artificial boundary condition on the one dimensional S wave function it is possible to compare the number of eigenstates of a non-local hamiltonian to the number of free eigenstates, provided the free and interacting eigenstates tend to coincide at high energies. Orthogonality of the wave functions, plus certain auxiliary hypotheses, enable one to prove the inequality $\delta(0) - \delta(\infty) \geqslant \nu\pi$, where ν is the number of bound states. At first sight, the completeness of the eigenstates of the non-local hamiltonian, which is proved for well behaved interactions, seems to be sufficient to establish the inequality $\delta(0) - \delta(\infty) \leqslant \nu\pi$ and, by comparison with the previous inequality, the theorem of Levinson, which says that the difference $\delta(0) - \delta(\infty)$ is equal to π times the number of bound states. However, one must take into account a possible degeneracy of the wave function for a positive energy. This fact, which never happens with a local regular potential, slightly modifies Levinson's theorem. It turns out that the difference $\delta(0) - \delta(\infty)$ is equal to π times the total number of eigenstates with vanishing asymptotic form, among which some states may have a positive energy. Application of these results to the Low equation is discussed.

1. – Introduction.

Levinson ([1]) has established a theorem connecting, for a given angular momentum, the number of bound states of a two body system, interacting by a local potential and the behaviour of the positive energy phase shifts. This

(*) Supported in part by the United States Air Force through the European Office Air Research and Development Command.

([1]) N. Levinson: *Dan. Mat. Fys. Med.*, **25**, 9 (1949).

theorem states that the difference $\delta(0) - \delta(\infty)$ is equal to π times the number of bound states. Several authors [2,3] have already given a formal proof of the validity of Levinson's theorem for hermitian interactions which are no longer diagonal in the coordinate representation. However it might be interesting to reexamine the problem for the following reasons:

(i) M. GOURDIN and the author [4] have shown that non-local separable interactions give the following relation for the phase shift instead of Levinson's theorem:

$$\delta(0) - \delta(\infty) = (\nu + \sigma)\pi\,, \tag{1}$$

ν is the number of bound states (0 or 1),
σ is the number of simple zeros of $\operatorname{tg}\delta$.

Levinson's theorem is satisfied provided $\sigma = 0$.

(ii) From an examination of the Low equation for neutral particles it turns out that any solution of Low's equation obeys the following condition:

$$\delta(0) - \delta(\infty) \leqslant \pi \tag{2}$$

and, when the scattering amplitude has σ zeros for positive energies,

$$-\pi(1+\sigma) \leqslant \delta(0) - \delta(\infty) \leqslant \pi(1-\sigma)\;; \tag{3}$$

these relations will be demonstrated in the last section.

Equation (2) is compatible with Levinson's theorem provided the system has no more than one bound state. On the other hand, if this condition is fulfilled, and if Levinson's theorem is taken seriously, equation (3) provides a selection of the solutions of Low's equation, which would be very useful if legitimate. Analogous attempts have already been made by HAAG [5], but instead of (2) and (3) he derives from the Low equation the relation:

$$\delta(0) - \delta(\infty) = (\nu - N)\pi\,,$$

where ν is the number of bound states and N is the total number of zeros of the scattering amplitude, including those lying in the non-physical range of negative energies. Unfortunately this relation did not clearly show that Le-

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(2) J. M. JAUCH: *Helv. Phys. Acta*, **30**, 143 (1957).
(3) B. S. DEWITT: *Lecture Notes*, Les Houches (1956).
(4) M. GOURDIN and A. MARTIN: *Nuovo Cimento*, **6**, 757 (1957).
(5) R. HAAG: *Nuovo Cimento*, **5**, 203 (1957).

vinson's theorem can never be satisfied by any solution of the Low equation if the system admits more than one bound state.

These two facts show that it might be worth-while to clarify the assumptions required to establish Levinson's theorem. Our method is as follows. We first count the positive energy interacting states after imposing an artificial boundary condition. We establish a correspondence between the interacting states and the free states which also are very easily counted. When the set of interacting states is complete, we have a one to one correspondence which provides a way of counting the negative energy interacting states, and therefore establishing Levinson's theorem. We show however that accidental degeneracies of the positive energy wave function introduce states which are omitted in the enumeration and lead to results of the type (1), in disagreement with Levinson's theorem.

2. – Notations. Basic and simplifying assumptions.

We restrict ourselves, for simplicity, to S states. Our treatment, however, might be easily extended to higher angular momenta. The reduced wave function is denoted by:

$$\psi(r) = r\Psi(r) \tag{4}$$

subject to the condition $\psi(0) = 0$; the non-local interaction, when explicitly needed, appears in the following way in the Schrödinger equation:

$$\frac{d^2}{dr^2}\psi(r) + E\,\psi(r) = \int V(r, r')\psi(r')dr' \tag{5}$$

with the convention $h^2/2Mr = 1$. (Notice a slight difference in notation with ref. (4)).

The operator $V(r, r')$ is taken to be hermitian to ensure the orthogonality of the wave functions. In fact it will be taken real, so that the hermiticity condition just becomes:

$$V(r, r') = V(r', r)\,. \tag{6}$$

We assume that the interaction vanishes at distances larger than r_0. More precisely $V(r, r') = 0$ when r or r' is larger than r_0. This assumption appreciably simplifies the demonstrations but it might be replaced by weaker conditions when r and r' go to infinity.

The wave function for positive energies has the asymptotic form $\psi(r) \sim \sim \sin(kr + \delta(k))$, $r > r_0$ and we make the assumption that the derivative of

the phase shift is never infinitely negative

$$\frac{\mathrm{d}\delta}{\mathrm{d}k} > -B\,,$$

where B is arbitrarily large. In fact, if we use explicitly an interaction of the type (6) with condition (7), this inequality is automatically satisfied, because (6) and (7) entail Wigner's inequality [6,7] on phase shifts:

$$\frac{\mathrm{d}\delta}{\mathrm{d}k} = -r_0 + \frac{\sin(kr_0+\delta)}{2k} + \int_0^{r_0}\psi^2(r)\,\mathrm{d}r > -r_0 + \frac{\sin 2(kr_0+\delta)}{dk}\,, \tag{9}$$

which, together with the assumption of a finite scattering length at zero energy guarantees condition (8). In the next two sections, however, the explicit form of the wave equation is not needed. The only important requirement is the mutual orthogonality of wave functions corresponding to different energies. For instance the non-relativistic kinetic energy operator might be replaced by a relativistic one without trouble.

Finally we assume that the interaction is sufficiently regular to give $\lim(\sin\delta(k)) = 0$. In this case, it is always permissible to make the convention $\delta(\infty) = 0$.

3. – Enumeration of the positive energy states.

We make the positive energy eigenstates discrete by imposing the boundary condition

$$\psi(R) = 0 \qquad (R > r_0), \tag{10}$$

i.e.

$$k_N R + \delta(k_N) = N\pi\,. \tag{10'}$$

Condition (10) does not modify the positive energy wave functions: since R is larger than the interaction range, it only selects certain states by (10′). The negative energy states, on the other hand, will be slightly modified by this condition. But it is expected that this change becomes vanishingly small as R goes to infinity.

Condition (8) enables us to choose R sufficiently large to be sure that $kR + \delta(k)$ is a monotonous, increasing function of k. Therefore, if $N_{\min}$ is the

(6) E. P. Wigner: *Phys. Rev.*, **98**, 145 (1955).
(7) A. Martin: *Compt. Rend.*, **243**, 22 (1956).

smaller possible value of N, the number of positive energy states with non vanishing asymptotic form will be, up to the state ψ_N:

$$N - N_{\min} + 1 .$$

Now we obtain $N_{\min}$ by taking k as small as possible. The smallest wave number k cannot be zero because at zero energy the asymptotic form of the wave function is $a+br$, where a, the scattering length, has been explicitly assumed to be finite. The behaviour of δ in the neighbourhood of zero energy is given by:

$$\delta(k) = n\pi + ak , \tag{11}$$

if

$$\delta(0) = n\pi .$$

Therefore, condition (10′) becomes:

$$k(R + a) \simeq (N_{\min} - n)\pi , \qquad k > 0. \tag{12}$$

$R+a$ is a positive quantity provided R is sufficiently large. The smallest possible value of N is $N_{\min} = n+1$, and the total number of positive energy states (with non-vanishing asymptotic form), up to the state ψ_N is $N - n$.

One notices that since $\delta(k) \to 0$ when $N \to \infty$, $k_N R/N \to \pi$ when $N \to \infty$. If we now consider the free states:

$$\varphi_N = \sin k'_N r \qquad \text{with} \qquad k'_N R = N\pi , \tag{13}$$

we see that $k_N - k'_N \to 0$ if $N \to \infty$. If the interaction is well behaved (i.e. non singular out of the origin and not too singular at the origin), the difference $\psi_N(r) - \varphi_N(r)$ will go to zero as N goes to infinity. On the other hand the number of free eigenstates up to φ_N is just N. If we can establish a connection between the total number of states φ up to N, and the total number of states ψ up to N, we may get some information on the number of negative energy states.

4. A theorem on two infinite sets of orthogonal functions and its consequences on the value of $\delta(0) - \delta(\infty)$.

Let us consider first a complete, orthonormal set:

$$\varphi_1 \varphi_2 \cdots \varphi_N \cdots$$

defined on a certain interval (in the present case $0 - R$).

Let us consider now a set of orthogonal functions, defined on the same interval, with the same boundary conditions, not necessarily complete, normalizable but not necessarily normalized to unity:

$$\psi_{A+1}\psi_{A+2}\cdots\psi_N\psi_{N+1}\cdots,$$

with the property that for N sufficiently large:

$$\psi_N(x)-\varphi_N(x)\to 0\,,$$

in the average, i.e.

$$\varepsilon_N=\int|\psi_N(x)-\varphi_N(x)|^2\,\mathrm{d}x\to 0\,. \tag{14}$$

The theorem says:

$$A\geqslant 0 \qquad \text{if} \qquad \sum_{i=1}^{\infty}\varepsilon_i \quad \text{converges.}$$

Roughly speaking, the number of states ψ is always smaller or equal to the number of states φ.

Proof: Since the set φ is a complete set we can expand any of the $N-A$ first states ψ_i in terms of the φ's.

$$\begin{cases}\psi_i=\sum_{i=1}^{\infty}a_{ij}\varphi_j\,,\\[2mm] a_{ij}=\int\varphi_j^*\psi_i\,\mathrm{d}x\,.\end{cases} \tag{15}$$

and therefore, if $i\neq j$

$$a_{ij}=\int(\varphi_j^*-\psi_j^*)\psi_i\,\mathrm{d}x\,,$$

by orthogonality of the ψ's.

By Schwarz's inequality:

$$|a_{ij}|^2\leqslant\int|\psi_i|^2\,\mathrm{d}x\int|\varphi_j-\psi_j|^2\,\mathrm{d}x=\varepsilon_j\int|\psi_j|^2\,\mathrm{d}x\,. \tag{16}$$

Let us now write (15) in the following way:

$$\psi_i-\sum_{j=N+1}^{\infty}a_{ij}\varphi_j=\sum_{j=1}^{N}a_{ij}\varphi_j \tag{17}$$

if $A<0$, the number of equations (17) is larger than N, and there necessarily

exists a linear relation between the left hand sides of equations (17),

$$\sum_{i=A}^{N} \mu_i \left(\psi_i - \sum_{j=N+1}^{\infty} a_{ij}\varphi_j\right) = 0 . \tag{18}$$

Hence, using orthogonality properties of the ψ's, and orthonormality of the φ's:

$$\sum_{i=A}^{N} |\mu_i|^2 \int |\psi_i|^2 \,\mathrm{d}x = \sum_{i=A}^{N} \sum_{j=N+1}^{\infty} |\mu_i|^2 |a_{ij}|^2 \tag{19}$$

and by (16)

$$\sum_{i=A}^{N} |\mu_i|^2 \int |\psi_i|^2 \,\mathrm{d}x \leqslant \sum_{i=A}^{N} \sum_{j=N+1}^{\infty} |\mu_i|^2 \varepsilon_j \int |\psi_i(x)|^2 \,\mathrm{d}x , \tag{20}$$

since all the terms of the series of the right hand side are positive, the order of summation can be inverted:

$$\sum_{i=A}^{N} |\mu_i|^2 \int |\psi_i|^2 \,\mathrm{d}x \leqslant \left[\sum_{i=A}^{N} |\mu_i|^2 \int |\psi_i|^2 \,\mathrm{d}x\right] \sum_{j=N+1}^{\infty} \varepsilon_j , \tag{20'}$$

hence

$$\boxed{1 \leqslant \sum_{j=N+1}^{\infty} \varepsilon_j .} \tag{21}$$

Therefore, if ε_j goes to zero sufficiently rapidly to make the series converge, one can take N sufficiently large to violate inequality (21). One concludes that the number of ψ's up to ψ_N is smaller or equal to the number of φ's up to φ_N, provided N is sufficiently large.

We can apply this theorem to our problem as follows: the states φ_N are the free states:

$$\varphi_N = \sin k'_N r , \qquad k'_N = \frac{N\pi}{R} ,$$

the states ψ_N are the states of the system; more precisely the states ψ_N are:

(i) the positive energy states with non-vanishing asymptotic form,

(ii) the ν negative energy states.

From the above theorem, we conclude that if $|\varphi_N - \psi_N|$ goes sufficiently rapidly to zero when N goes to infinity,

$$\nu + N - m \leqslant N ,$$

therefore

$$\delta(0) - \delta(\infty) \geqslant \pi\nu \,, \tag{22}$$

where ν is the number of bound states. This inequality is a generalization of Levinson's theorem. The condition on the convergence of the series $\sum \varepsilon_n$ is generally satisfied by well behaved interactions. In particular it is fulfilled in the case of a local potential vanishing at $r = r_0$ and less singular than $1/r^2$ at the origin. It is also satisfied by not too singular separable potentials, $V(r, r') = \lambda u(r) u(r')$. In this case the wave function ψ reads, in co-ordinate space:

$$\psi(r) - \sin kr = \sin kr \left[\frac{\lambda}{k}\int\!\!\int u(r')G(r', r'')u(r'')\,\mathrm{d}r'\,\mathrm{d}r''\right] - \\ - \frac{\lambda}{k}\int_0^\infty G(r, r')u(r')\,\mathrm{d}r' \cdot \int_0^\infty \sin kr'' u(r'')\,\mathrm{d}r'' \,, \tag{23}$$

where $G(r', r'') = \sin(kr<) \cos(kr>)$.

It is easily seen that if $u(r)$ behaves like $1/r$ in the neighbourhood of the origin, the modulus of the right hand side of (23) does not exceed $Ck^{2\beta-3}$ for sufficiently large k. Consequently $\psi_N - \varphi_N \to 0$ if $\beta < \frac{3}{2}$ (this condition is in fact the condition of existence of ψ ([4])), the convergence of the series $\sum_N^\infty \varepsilon_n$ is guaranteed if

$$\int_0^R |\psi_N - \varphi_N|^2\,\mathrm{d}r \leqslant \frac{\text{const}}{N^{1+\eta}} \,,$$

which implies

$$\beta < \frac{5}{4} \,. \tag{24}$$

Inequality (24) must be compared with the condition for the existence of a solution of the Schrödinger equation, which is $\beta < \frac{3}{2}$. In the case of a general non-local potential one can derive a necessary condition—which might well be sufficient too—for the convergency of the series $\sum \varepsilon_n$: we split $\sum \varepsilon_n$ in two parts:

$$\sum_N \varepsilon_N = \sum_N \int_0^{r_0} |\psi_N - \varphi_N|^2\,\mathrm{d}r + \sum_N \int_0^R |\psi_N - \varphi_N|^2\,\mathrm{d}r \,;$$

these two parts must be separately convergent,

$$\int_{r_0}^{R} |\psi_N - \varphi_N|^2 \,\mathrm{d}r = \int_{r_0}^{R} |\sin(k_N r + \delta) - \sin(k'_N r)|^2 \,\mathrm{d}r =$$

$$= 4\int_{r_0}^{R} \sin^2\left[\frac{(k_N - k'_N)r + \delta}{2}\right] \cos^2\left[(k_N + k'_N)r + \delta\right] \mathrm{d}r \,.$$

For k_N sufficiently large, the cosine term oscillates so rapidly that it can be replaced by its average value, and, finally, taking into account (10′) and (13), one gets:

$$\int_{r_0}^{R} |\psi_N - \varphi_N|^2 \,\mathrm{d}r \simeq (R - r_0) - \frac{R}{\delta}\sin\left[\frac{\delta}{R}(R - r_0)\right] \simeq \frac{1}{6}\frac{\delta^2}{R^2}(R - r_0)^3 \quad \text{if } \delta \to 0, \tag{25}$$

the convergence of the series requires:

$$\delta \sim \frac{1}{k^\alpha}, \qquad \alpha > \frac{1}{2}\,. \tag{26}$$

If the Born approximation is valid at high energies:

$$\delta \sim \mathrm{const}\frac{1}{k}\int \sin kr\, V(r, r') \sin kr' \,\mathrm{d}r\,\mathrm{d}r' = \mathrm{const}\frac{1}{k}\, V(k, k)\,,$$

so that the condition becomes:

$$V(k, k) \sim k^\gamma, \qquad \gamma < \frac{1}{2}, \qquad \text{for } k \to \infty\,. \tag{27}$$

This condition is probably sufficient, because it is expected that for sufficiently large R, the contribution from r_0 to R, in ε_n, largely dominates the contribution from 0 to r_0. Besides, it can be seen that (27) coincides with (24) for separable interactions.

About the behaviour of the interaction at large distances we cannot say much because we explicitly assume that the interaction vanishes for $r > r_0$. But it is very likely that this very stringent condition is unnecessary and can be replaced by weaker requirements. This can be rather easily done for local interactions; then a small quantity must be added to $\delta(k)$ in condition (10′). In the non-local case, however, the whole wave function is affected by the condition $\psi(R) = 0$ inside the range of the interaction.

Finally it may be asked if the expansion of the ψ's in terms of the φ's is legitimate, since the completeness of a system of functions φ is only valid for a certain class of functions ψ. This point will be touched upon in the next section.

5. - Completeness of the eigenfunctions of a non local hamiltonian.

If one can show that the system of the ψ's is also a complete system, the roles of φ and ψ can be exchanged, and then it can be shown that the number of eigenstates of the hamiltonian up to ψ_N is certainly larger or equal to the number of free states up to φ_N, i.e. larger or equal to N. If one then assumes that there does not exist any other eigenstate than the $N-m$ positive energy ψ_N and the ν bound states, one gets $\delta(0)-\delta(\infty)\leqslant\nu\pi$, which, when compared with (22), gives the theorem of Levinson $\delta(0)-\delta(\infty)=\nu\pi$.

We shall only outline the demonstration of the completeness of the set of states ψ_N such that $\psi_N(0)=\psi_N(R)=0$, obeying the Schrödinger equation (5). Our demonstration is largely inspired by the demonstration given for the case of a local interaction by MORSE and FESHBACH [8]. First one shows the equivalency of equation (5) with the variational principle:

$$\delta\,\frac{\int\psi(r)H(r,r')\psi(r')\,\mathrm{d}r\,\mathrm{d}r'}{\int\psi(r)\psi(r)\,\mathrm{d}r}=0\,.$$

One assumes that H admits a lower eigenvalue E_1 and that the eigenvalue E_N corresponding to the N-th eigenstate can be made arbitrarily large by taking N large enough.

Then one demonstrates that if f is any function orthogonal to the $k-1$ first eigenstates $\psi_1,\ \psi_2,\ldots,\ \psi_{k-1}$,

$$\frac{\int_0^R f(r)\big(H(r,r')-E_1\big)f(r)\,\mathrm{d}r\,\mathrm{d}r'}{\int_0^R[f(r)]^2\,\mathrm{d}r}\geqslant E_k-E_1\,.$$

This inequality is applied to the quantity:

$$f=F-\sum_{i=1}^{k-1}C_i\psi_i\qquad\text{where}\qquad C_i=\int_0^R\psi_iF\,\mathrm{d}r\,,$$

(8) P. MORSE and H. FESHBACH: *Mathematical Methods of Theoretical Physics* (New York, 1953), p. 738.

which is orthogonal to the $k-1$ first eigenfunctions, where $F(r)$ is any function vanishing for $r=0$ and $r=R$. This gives (useless symbols have been omitted):

$$\int |F - \sum_1^{k-1} C_i \psi_i|^2 \frac{\int F(H-E_1)F}{E_k - E_1} . \tag{28}$$

If the fixed quantity $\int F(H-E_1)F$ is finite, the left hand side can be made arbitrarily small by choosing k large enough. The completeness of the set ψ for functions F, such that $\int F(H-E_1)F=$ finite quantity, is established. To study the possibility of expanding the φ's in terms of the ψ's we just have to look if the quantity

$$\int_0^R \sin krH(r,r')\sin kr'\,\mathrm{d}r\,\mathrm{d}r' = k^2\int_0^R \sin^2 kr\,\mathrm{d}r +$$
$$+\int_0^R\int_0^R \sin krV(r,r')\sin kr'\,\mathrm{d}r\,\mathrm{d}r' = k^2\int_0^R\sin^2 kr\,\mathrm{d}r + \text{const } V(k,k)\,,$$

is finite for any value of k. This condition is fulfilled by acceptable non-local interactions everywhere regular except perhaps at $r=0$ or $r'=0$. Otherwise, the behaviour of the integrand for small r and r' being just given by $rV(r,r')r'$, the interaction in momentum space $V(p,p')$ would be always infinite, for any p and any p'.

Conversely one might ask: is it possible, as was done in Sect. **2**, to expand the ψ's in terms of the φ's? Then one must show that

$$\int_0^R \psi\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right)\psi\,\mathrm{d}r$$

is finite. Since $V(r,r')$ is regular everywhere except perhaps at the origin, the quantity $\psi(\mathrm{d}^2/\mathrm{d}r^2)\psi$ is always finite, except perhaps at the origin.

If $\int\psi(-\mathrm{d}^2/\mathrm{d}r^2)\psi\,\mathrm{d}r$ is infinite, this means that in the expression:

$$E_N = \frac{\int_0^R \psi_N(-\mathrm{d}^2/\mathrm{d}r^2)\,\psi_N\,\mathrm{d}r + \int_0^R\int_0^R \psi_N(r)V(r,r')\psi_N(r')\,\mathrm{d}r\,\mathrm{d}'r'}{\int_0^R[\psi_N]^2\,\mathrm{d}r}\,;$$

there is a cancellation between two infinite quantities. This fact never happens with acceptable non-local interactions. We notice that the finiteness of the above integral gives $\psi\sim r^\alpha$, $\alpha>\frac{1}{2}$ near the origin.

617

From this we conclude that when one assumes an interaction $V(r, r')$, vanishing at $r = r_0$ and that $|\psi_N - \varphi_N|$ tends sufficiently rapidly to zero when N goes to infinity, the number of eigenstates ψ of H up to ψ_N is equal to the number of free eigenstates φ up to φ_N, and if the $N - m$ positive energy states are the only positive energy eigenstates of H up to ψ_N, the theorem of Levinson is proved for a non-local interaction.

6. – Exceptions to Levinson's theorem.

One might now ask why there exist exceptions to Levinson's theorem, with non-local interactions satisfying all the conditions above mentioned ([4]). Since, according to Sect. **5**, the completeness of the hamiltonian is proved, the only possibility is that, in the list of the positive energy states, certain states are missing. All the above considerations implicitely assume that for every positive energy k^2 there exists one wave function ψ_N with one asymptotic form $\sin(kr + \delta(k))$. This is certainly true for a local interaction everywhere finite (except perhaps at the origin), because, in this case, general theorems on the solutions of second order differential equations guarantee the uniqueness of the solution, and, on the other hand, the wave function and its first derivative can never vanish at the same time, so that the asymptotic form certainly exists.

In the case of non-local interactions the situation is not so simple, and we shall investigate the possibility of a degeneracy of the wave function for a positive energy. Let us write the integral equation equivalent to the Schrödinger equation:

$$\psi(r) = \sin kr - \frac{1}{k}\int G(r, r')V(r', r'')\psi(r'')\,\mathrm{d}r'\,\mathrm{d}r'' , \tag{29}$$

where

$$G(r, r') = \sin(kr <)\cos(kr >) .$$

If there is a degeneracy, the homogeneous equation:

$$\chi(r) = -\frac{1}{k}\int G(r, r')V(r', r'')\chi(r'')\,\mathrm{d}r'\,\mathrm{d}r'' , \tag{30}$$

must simultaneously have a solution. The general solution is then $\psi_0(r) + A\chi(r)$. Generally, when (30) admits a solution, (29) has no solution, and this just means that the phase shift goes through $\pi/2 + k\pi$, since the asymptotic form is then

$$\chi(r) \sim -\frac{\cos kr}{k}\int_0^\infty \sin kr'\, V(r', r'')\chi(r'')\,\mathrm{d}r'\,\mathrm{d}r'' . \tag{31}$$

It is easy to get a necessary condition for the compatibility of equations (29) and (30). Multiplying (29) by $\int\chi(r''')V(r''',r)$ and (30) by $\int\psi(r''')V(r''',r)$, integrating over r and r''', subtracting the two equations thereby obtained, and taking into account hermiticity of the interaction, one gets:

$$\int \sin kr\, V(r,r')\chi(r')\,\mathrm{d}r\,\mathrm{d}r' = 0\,. \tag{32}$$

Comparing (32) and (31) we see immediately that in this case, the asymptotic form of $\chi(r)$ vanishes at large distances. More precisely, comparing (30) and (32) we see that:

$$\chi(r) = -\frac{1}{k}\sin kr\int_r^\infty \cos kr'\, V(r',r'')\chi(r'')\,\mathrm{d}r'\,\mathrm{d}r''\,,$$

$$+\frac{1}{k}\cos kr\int_r^\infty \sin kr'\, V(r',r'')\chi(r'')\,\mathrm{d}r'\,\mathrm{d}r''\,,$$

vanishes for r larger than the range of the interaction r_0. Consequently, while $\psi(r)$ is not unique, the asymptotic form of the solution is unique. This is the reason why such a phenomenon does not appear when one merely computes the phase shift but not the wave function.

We shall now try to answer the question: assuming that (30) has a solution, is condition (32) sufficient to guarantee that (29) has a solution too? Let us consider the equation:

$$\chi(r) = -\frac{\lambda}{k}\int G(r,r')V(r',r'')\chi(r'')\,\mathrm{d}r'\,\mathrm{d}r''\,,\quad \text{(for fixed } k\text{)}, \tag{33}$$

Its kernel is not symmetrical, but its eigenfunctions have nevertheless a kind of generalized orthogonality property:

$$\int \chi_\lambda(r')V(r',r'')\chi_{\lambda'}(r'')\,\mathrm{d}r'\,\mathrm{d}r'' = \delta_{\lambda\lambda'}\,. \tag{34}$$

Any function f can be tentatively expanded as:

$$f(r) = \sum_{\lambda_i}\chi_{\lambda_i}(r)\int f(r')V(r',r'')\chi_{\lambda_i}(r'')\,\mathrm{d}r'\,\mathrm{d}r''. \tag{35}$$

We therefore try to find a solution of (29) of the following form

$$\psi(r) = \sum a_{\lambda_i}\chi_{\lambda_i}(r)\,. \tag{36}$$

Carrying (36) in (29) one gets:

$$\sum_i a_{\lambda_i}\chi_{\lambda_i}(r) = \sin kr + \sum_i \lambda_i a_{\lambda_i}\chi_{\lambda_i}(r)\,,$$

and, then, using (34):

$$a_{\lambda_i} = \int \chi_{\lambda_i}(r)V(r, r') \sin \mathrm{k}r' \,\mathrm{d}r\,\mathrm{d}r' + \lambda_i a_i\,,$$

or

$$a_{\lambda_i} = \frac{\int \chi_{\lambda_i}(r)V(r, r') \sin kr' \,\mathrm{d}r\,\mathrm{d}r'}{1-\lambda_i}\,.$$

Since (30) is satisfied, one of the λ_i's is equal to unity, but the equation for a_i is automatically satisfied because of (32). The solution then reads:

$$\psi(r) = A\chi(r) + \sum_{\lambda_i \neq 1} a_{\lambda_i}\chi_{\lambda_i}(r)\,.$$

The weak point in this demonstration is that we have not proved the « completeness » of the set χ'.

This study shows that whenever the homogeneous equation admits a solution with a vanishing asymptotic form, the inhomogeneous equation (29) admits a one-parameter set of solutions. We notice that condition $\psi(R) = 0$ imposed in Sect. **3** is always automatically satisfied by the solution of (30) obeying condition (32). In momentum space, condition (32) reads:

$$\int V(k, p)\chi(p)\,\mathrm{d}p = 0\,, \tag{37}$$

where $\chi(p)$ is a solution of the homogeneous equation

$$\chi(p) = \frac{1}{p^2 - k^2}\int V(p, p')\chi(p')\,\mathrm{d}p'. \tag{38}$$

We can say that a degeneracy happens whenever the homogeneous Schrödinger equation in momentum space admits a solution which is regular for $p = k$.

Let us now call ν' the total number of states χ. We have proved in Sect. **5** that, up to ψ_N, the total number of linearly independent eigenstates of H was N. We get the following equation:

$$N = N - m + \nu + \nu' \quad \text{and} \quad \delta(0) - \delta(\infty) = \pi(\nu + \nu')\,. \tag{39}$$

Levinson's theorem must be modified as follows: the difference $\delta(0) - \delta(\infty)$ is equal to π times the total number of states ψ such that $\psi(\infty) = 0$, with negative (bound states) or positive energy (states χ).

7. – Examples of violations of Levinson's theorem.

As mentioned in Sect. **1**, the first example of the situation described in the preceding section was found by M. GOURDIN and the author [4] in a study of non-local separable interactions. In this case:

$$\operatorname{tg}\delta = -\frac{\pi}{2k}\,\frac{f(k)}{1+\lambda\int\limits^{\infty} f(p)\,\mathrm{d}p/(p^2-k^2)}, \tag{40}$$

where

$$f(k) = [v(k)]^2,$$

and $v(k)$, apart from trivial factors, is the sine transform of $v(r)$ appearing in the *reduced* interaction:

$$V(r, r') = v(r)\,v(r').$$

The above studied situation appears when, simultaneously

$$f(k) \qquad \text{and} \qquad 1+\lambda\int\limits_0^{\infty}\frac{f(p)\,\mathrm{d}p}{p^2-k^2},$$

vanish. In this case, the general solution of the Schrödinger equation in momentum space is given by:

$$\delta(p-k) + A\,\frac{v(p)}{p^2-k^2}. \tag{41}$$

All these solutions give the same phase shift $\delta = 0$.

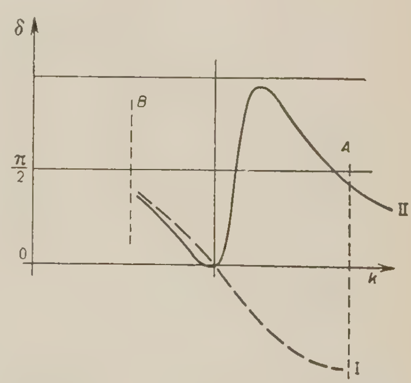

Fig. 1. – I). Exact coincidence of the zeros of the numerator and of the denominator in $\operatorname{tg}\delta$; II). Approximate coincidence.

On Fig. 1, we have drawn the curves giving the phase shift for the critical value of λ(I) and for a slightly different value (II). These curves show us clearly how one state with non-vanishing asymptotic form disappears in situation I. Indeed

$$(kR+\delta)_{A\,\mathrm{II}} - (kR+\delta)_{B\,\mathrm{II}} \simeq \pi + (kR+\delta)_{A\,\mathrm{I}} - (kR+\delta)_{B\,\mathrm{I}}.$$

The missing state is replaced by the state $\chi(p) = u(p)/(p^2 - k^2)$. In the present case, condition (32) reduces to

$$\int v(r) \sin kr \, \mathrm{d}r = 0 \qquad \text{or} \qquad v(k) = 0 \, . \tag{42}$$

We shall now give another example, which seems more convincing because, in this case, the integral $\int \sin kr\, V(r, r')\, \mathrm{d}r$ does not vanish, while $\int \sin kr V(r, r') \cdot \cdot \varphi(r')\, \mathrm{d}r\, \mathrm{d}r' = 0$. We just take an interaction which is the sum of two separable interactions. The inhomogeneous and homogeneous integral equations in momentum space read:

$$\psi(p) = \delta(p - k) - \frac{1}{p^2 - k^2} \int [\lambda f(p) f(p') + \mu g(p) g(p')] \psi(p')\, \mathrm{d}p' , \tag{43}$$

$$\chi(p) = - \frac{1}{p^2 - k^2} \int [\lambda f(p) f(p') + \mu g(p) g(p')] \chi(p')\, \mathrm{d}p' \, . \tag{43$'$}$$

We first try to adjust λ and μ in order to insure the existence of a solution of the homogeneous equation which is regular for $p = k$. Following the classical procedure we write

$$\chi(p) = - \frac{1}{p^2 - k^2} [\lambda C_1 f(p) + \mu C_2 g(p)] ,$$

$$C_1 = \int f(p) \chi(p)\, \mathrm{d}p , \qquad C_2 = \int g(p) \chi(p)\, \mathrm{d}p ,$$

and we use the notation

$$[xy] = P \int \frac{x(p) y(p)}{p^2 - k^2}\, \mathrm{d}p \, .$$

The existence of a solution gives the system of equations:

$$\begin{cases} C_1 = - \lambda [f^2] C_1 + \mu [fg] C_2 , \\ C_2 = \lambda [fg] C_1 + \mu [g^2] C_2 \, . \end{cases} \tag{44}$$

The regularity condition gives:

$$\lambda f(k) C_1 + \mu g(k) C_2 = 0 \, . \tag{45}$$

Combining (44) and (45) we obtain the necessary and sufficient condition

for the existence of a regular solution:

$$(46)\qquad \begin{cases} g(k) = \lambda[f(k)[fg] - g(k)[f^2]] \,, \\ f(k) = \mu[g(k)[fg] - f(k)\,[g^2] \,, \end{cases}$$

and the corresponding expression for χ is just

$$(47)\qquad \chi(p) = \frac{1}{p^2 - k^2}\,[f(k)g(p) - f(p)g(k)] \,.$$

We shall now check that conditions (46) guarantee the existence of a solution of the inhomogeneous equation (43′). This solution, if it exists, would be:

$$(48)\qquad \psi(p) = \delta(p - k) - \frac{1}{p^2 - k^2}\,[\lambda f(p)\Gamma_1 + \mu g(p)\Gamma_2] \,,$$

with the conditions

$$(49)\qquad \Gamma_1 = f(k) - \lambda[f^2]\,\Gamma_1 - \mu[fg]\Gamma_2 \,, \qquad \Gamma_2 = g(k) - \lambda\,[fg]\Gamma_1 - \mu\,[g^2]\Gamma_2 \,.$$

It is straightforward to see, by mere substitution of λ and μ taken from (46), that the two equations (49) are proportional to each other. We can take for instance $\Gamma_1 = 0$. One then gets the solution:

$$(50)\qquad \psi(p) = \delta(p - k) - \frac{f(k)g(p)}{(p^2 - k^2)[fg]} \,,$$

and with $\Gamma_2 = 0$ one gets

$$(50')\qquad \psi(p) = \delta(p - k) - \frac{f(p)g(k)}{(p^2 - k^2)[fg]} \,.$$

Anyhow, in both cases, the phase shift δ_0 is given by:

$$(51)\qquad \operatorname{tg}\delta_0 = \frac{\pi}{2k}\,\frac{f(k)g(k)}{\int (f(p)g(p)\,\mathrm{d}p)/(p^2 - k^2)} \,.$$

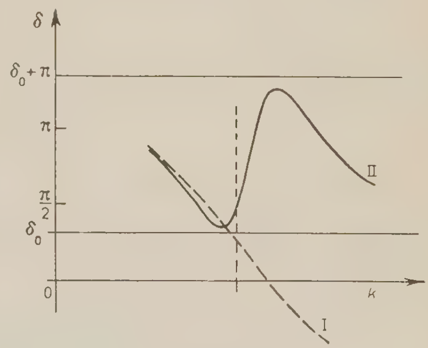

Fig. 2.

If the constants λ and μ are slightly different from their values given by (49), the behaviour of the phase shift is easily seen to be described by curve II on Fig. 2 (curve I corresponds to the case when (46) is rigorously satisfied).

623

Finally we give a last case where (29) and (30) have simultaneous solutions. It is the trivial case of a local potential which is infinite somewhere, so that the wave function is split in two independent parts. Most of the time the internal wave function is zero, but, for the eigenvalues of the internal potential, the wave function reads:

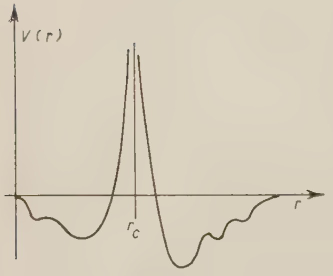

Fig. 3.

$$\psi_{\text{external}} + A\psi_{\text{internal}} \, .$$

One can check that condition (32) is satisfied by ψ_{internal} because

$$\int_0^{r_c} \sin kr \, V(r)\, \psi_{\text{int}}(r)\, \mathrm{d}r = \int_0^{r_c} \sin kr \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2\right] \psi_{\text{in}}(r)\, \mathrm{d}r = 0 \, .$$

In this case, $\delta(k)$ reaches no limit when $k \to \infty$ because the internal potential has an infinite number of eigenstates.

8. – Compatibility of the solutions of Low equation with the inequality $\delta(0) - \delta(\infty) \geqslant \nu\pi$.

This question has already been investigated by R. HAAG [5]. We wish to add, nevertheless, a few remarks. Consider, for instance, the real part of the relativistic uncoupled Low equation

$$\text{(52)} \qquad \operatorname{Re}\,(h^+(\omega)) = \sum A_i \left[\frac{1}{\omega_i - \omega} + \frac{1}{\omega_i + \omega}\right] + \frac{P}{\pi} \int_\mu^\infty v(k')k' \, |h^+(\omega)'|^2 \left[\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega}\right] \mathrm{d}\omega' \, ,$$

where

$$h^+(\omega) = \frac{\exp[i\delta] \sin \delta}{kv(k)} \, .$$

Let us compute the derivative of $\operatorname{Re}\,(h^+(\omega))$ when $h^+(\omega) = 0$. Making use of the fact that:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} P \int \frac{\varphi(\omega)\, \mathrm{d}\omega}{\omega - z}\right) = \int \frac{\varphi(\omega)\, \mathrm{d}\omega}{(\omega - z_0)^2} \, , \qquad z = z_0$$

if

$$\varphi(z_0) = \varphi'(z_0) = 0 \,,$$

one sees immediately that

$$\text{(53)} \qquad \frac{\mathrm{d}\,\mathrm{Re}\,(h^+(\omega))}{\mathrm{d}\omega} = \sum A_i \left[\frac{1}{(\omega-\omega_i)^2} - \frac{1}{(\omega+\omega_i)^2}\right] +$$

$$+ \frac{P}{\pi}\int_\mu^\infty v(k')k'\,|h^+(\omega')|^2 \left[\frac{1}{(\omega'-\omega)^2} - \frac{1}{(\omega'+\omega)^2}\right] \mathrm{d}\omega' \qquad \text{for} \qquad h^+(\omega)=0 \,.$$

now

$$\text{(53)} \qquad \frac{\mathrm{d}\,\mathrm{Re}\,(h^+(\omega))}{\mathrm{d}\omega} = \frac{1}{kr(k)}\frac{\mathrm{d}\delta}{\mathrm{d}\omega}\,,$$

when $h^+(\omega)=0$.

In other words, when the scattering amplitude vanishes, the derivative of the phase shift is positive. This trivial remark enables us to draw the dependence of the phase shift on the energy (Fig. 4).

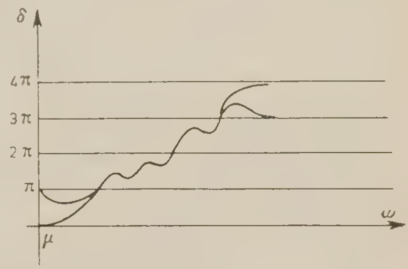

Fig. 4.

Fig. 4 shows clearly that if N^+ is the number of zeros of the scattering amplitude in the physical range of energies,

$$\text{(54)} \qquad \delta(0)-\delta(\infty) = \begin{cases} -(N^+-1)\pi \\ -N^+\pi \\ -(N^++1)\pi \end{cases}$$

In the relation given by Haag (5),

$$\delta(0)-\delta(\infty) = \pi(\nu - N)\,,$$

N was the total number of zeros of the scattering amplitude, including those appearing in the non-physical range.

Equations (54) show that $\delta(0)-\delta(\infty) \leqslant \pi$. Therefore, if the number of bound states is larger than one, there does not exist any solution of the Low equation satisfying condition $\delta(0)-\delta(\infty) \geqslant \nu\pi$. We notice that in practical cases, there never appears more than one bound state in the Low equation. If there exists one or zero bound state, this condition gives $N^+=0$ or 1. Evidently, for the selection of the solution, the relation derived by HAAG is more powerful than relation (54) since it gives $N^+=0$.

The question arises: is the use of Levinson's theorem or of the less stringent inequality derived in Sect. **4** permissible in the present case? If it is possible to represent the meson interacting with the nucleon field by a wave function, two wave functions belonging to different energies being orthogonal to each other, then the application of the inequality $\delta(0) - \delta(\infty) > \nu\pi$ is legitimate.

However, it is not permissible to say that this is the case in a field theoretical approach to meson-nucleon scattering. The one meson amplitudes which are the only amplitudes with non-vanishing asymptotic form play the role of the wave functions. These amplitudes are not mutually orthogonal, because the states of the system contain many meson amplitudes (with vanishing asymptotic form). When these many meson amplitudes can be neglected, one can show, from the orthogonality of the total states, that the one meson wave functions are approximately orthogonal; this inequality $\delta(0) - \delta(\infty) > \nu\pi$ can then be used to select the solutions of Low's equation and leads to the « normal » solution, which is in agreement with the known fact that this « normal » solution coincides with the solution obtained by a perturbation expansion. On the other hand, if the many meson amplitudes play an important role, we cannot draw any conclusion. An extreme picture of this situation is proposed by DYSON [9] who gives a model in which the states are mutually orthogonal, but the scattered waves are not, so that Levinson's theorem breaks down.

9. – Concluding remarks.

We have shown that orthogonality of the wave functions corresponding to different energies is the main condition for the validity of the inequality $\delta(0) - \delta(\infty) > \nu\pi$, where ν is the number of bound states of the system. The completeness of the set of eigenstates of the Hamiltonian, which is generally fulfilled by non-local interactions, is not sufficient to prove that $\delta(0) - \delta(\infty) \leqslant \nu\pi$ and hence $\delta(0) - \delta(\infty) = \nu\pi$. Levinson's theorem is only valid provided there is no degeneracy for positive energy wave functions. These degeneracies, when happening, do not affect the asymptotic form of the wave function, and, to make the set of eigenstates of the Hamiltonian complete, one must take into account a series of ν' positive energy states with a vanishing asymptotic form. Levinson's theorem becomes

$$\delta(0) - \delta(\infty) = \pi(\nu + \nu') .$$

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(9) F. J. DYSON: *Phys. Rev.*, **106**, 157 (1957).

Here $\nu - \nu'$ is the total number of linearly independent eigenstates with vanishing asymptotic form.

The inequality $\delta(0) - \delta(\infty) \geqslant \nu\pi$ gives some indication in favour of the « normal » solution of the Low equation, but no definite conclusion can be drawn in the field theoretical case.

RIASSUNTO (*)

Introducendo una artificiosa condizione al limite per la funzione d'onda S unidimensionale, è possibile confrontare il numero degli autostati di una hamiltoniana non locale col numero di autostati liberi, purchè gli autostati liberi ed interagenti tendano a coincidere alle alte energie. L'ortogonalità delle funzioni d'onda insieme ad alcune ipotesi ausiliarie ci mette in condizione di provare l'ineguaglianza $\delta(0) - \delta(\infty) \geqslant \nu\pi$, dove ν è il numero di stati legati. A prima vista la completezza degli autostati dell'hamiltoniana non locale, che è provata per le interazioni regolari, sembra essere sufficiente a stabilire l'ineguaglianza $\delta(0) - \delta(\infty) \leqslant \nu\pi$ e, per confronto con la precedente ineguaglianza, il teorema di Levinson che afferma che la differenza $\delta(0) - \delta(\infty)$ è uguale a π volte il numero degli stati legati. Tuttavia, si deve tener conto di una possibile degenerazione della funzione d'onda per energia positiva. Questa circostanza, che non si verifica mai con un potenziale locale regolare, modifica leggermente il teorema di Levinson. Risulta che la differenza $\delta(0) - \delta(\infty)$ è uguale a π volte il numero totale degli autostati con forma asintoticamente tendente a zero, tra i quali qualche autostato può possedere energia positiva. Si discute l'applicazione di questi risultati all'equazione di Low.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Zum Energie-Impulstensor des elektromagnetischen Feldes im bewegten Dielektrikum.

J. I. Horváth

Institut für Theoretische Physik der Universität - Szeged

(ricevuto il 22 Ottobre 1957)

Zusammenfassung. — Es wird auf Grund des Giorgi-Sommerfeldschen Systems bewiesen, daß die Konstruktion des Energie-Impulstensors des elektromagnetischen Feldes auch im Vakuum mit dem Minkowskischen Ansatz übereinstimmt. Da die Formulierung der Theorie dimensionsinvariant sein soll, wird darauf geschlossen, daß der Minkowskische Energie-Impulstensor des Feldes als der richtige bezeichnet werden kann.

1. – Einleitung.

Die phänomenologische Theorie der relativistischen Elektrodynamik in den Dielektriken ist eine der wichtigsten und interessantesten Fragen der elektromagnetischen Feldtheorie, die aber auch heute mehrere umstrittene und offene Probleme aufweist. Unter diesen Problemen ist unzweifelhaft dasjenige das interessanteste, das mit den Schwierigkeiten des Minkowskischen Energie-Impulstensors im Zusammenhang steht und das vor kurzer Zeit von mehreren Forschern von verschiedenen Gesichtspunkten aus neubehandelt wurde (1-12).

(1) N. L. Balázs: *Phys. Rev.*, **91**, 408 (1953).
(2) F. Beck: *Zeits. f. Phys.*, **134**, 580 (1953).
(3) G. Györgyi: *Acta Phys. Acad. Sci. Hung.*, **4**, 121 (1954).
(4) J. I. Horváth: *Bull. Acad. Pol. Sci. Cl. III*, **4**, 447 (1956).
(5) M. v. Laue: *Zeits. f. Phys.*, **128**, 387 (1950).
(6) G. Marx und G. Györgyi: *Acta Phys. Acad. Sci. Hung.*, **3**, 213 (1954).
(7) G. Marx und K. Nagy: *Acta Phys. Acad. Sci. Hung.*, **4**, 213 (1955).
(8) K. F. Novobátzky: *Hung. Acta Phys.*, **1**, No. 5, 23 (1949).
(9) C. Møller: *The Theory of Relativity* (Oxford, 1952).
(10) A. Sommerfeld: *Elektrodynamik* (Leipzig, 1949).
(11) A. Rubinowicz: *Acta Phys. Pol.*, **14**, 225 (1955).
(12) I. G. Tamm: *Journ. of Phys. USSR*, **1**, 439 (1939).

Im Anschluß and diese interessanten Untersuchungen möchten wir dadurch einen neuen Gesichtspunkt aufzuwerfen, daß wir die relativistische Elektrodynamik im Dielektrikum auf Grund des Giorgi-Sommerfeldschen Systems entwickeln wollen. Wir werden in dieser Arbeit die beiden rivialisierenden Energie-Impulstensore, d.h. den Minkowskischen, sowie den Abrahamschen behandeln, dann wollen wir darauf hinweisen, daß sich die beiden Tensoren im Vakuum auf den mit Hilfe des Minkowskischen Ansatzes konstruierten Energie-Impulstensor des Feldes reduzieren. Da die Formulierung der Theorie dimensionsinvariant sein soll, können wir daraus auf diese Weise darauf schließen, daß der Minkowskische Energie-Impulstensor des Feldes als der richtige bezeichnet werden kann.

2. – Die Grundgleichungen der relativistischen Elektrodynamik.

Die relativistische Theorie des elektrodynamischen Feldes wurde schon in dem wohlbekannten Lehrbuch von A. Sommerfeld ([10]) auf Grund des Giorgi-Sommerfeldschen Systems entwickelt. Wir haben demnoch einige wichtige Zusammenhänge der Theorie jetzt kurz zu wiederholen, denn wir wollen den pseudo-euklidischen Raum mit dem metrischen Grundtensor

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1\,, \qquad g_{\mu\nu} = 0 \quad (\mu \neq \nu) \tag{2.1}$$

als die geometrische Basis der Theorie wählen.

Die Komponenten des Viererpotentials werden durch

$$\boldsymbol{\Phi} = \{\Phi_\mu\} = \{-\sqrt{\varepsilon_0\mu_0}\,\varphi, \mathfrak{A}\} = \left\{-\frac{1}{c}\varphi, \mathfrak{A}\right\}, \tag{2.2}$$

definiert, wo φ das skalare, $\mathfrak{A}$ das Vektorpotential, ε_0 die Dielektriszitätkonstante, bzw. μ_0 die magnetische Permeabilität des Vakuums und c die Lichtgeschwindigkeit im Vakuum bedeuten. Es ist bekannt, daß

$$c = (\varepsilon_0\mu_0)^{-\frac{1}{2}} \tag{2.3}$$

ist. Das Viererpotential soll der Wellengleichung

$$\Box\boldsymbol{\Phi} = \mu_0\mathbf{s} \tag{2.4}$$

genügen, wo

$$\Box \overset{\text{def}}{=} g^{\alpha\beta}\partial_\alpha\partial_\beta \equiv \varepsilon_0\mu_0\,\partial_t^2 - \Delta\,,$$

und

$$s = \{s_\alpha\} = \{-\varrho/\sqrt{\varepsilon_0\mu_0},\ i\} = \{-c\varrho,\ i\} \tag{2.5}$$

bedeuten und ϱ die Ladungsdichte, bzw. i die Stromdichte bedeutet.

Die Intenstitätsgrößen des elektromagnetischen Feldes, d.h. die elektrische Feldstärke $\mathfrak{E}$ und der Induktionsvektor $\mathfrak{B}$ werden durch den Feldtensor

$$F_{\alpha\beta} = \partial_\alpha \Phi_\beta - \partial_\beta \Phi_\alpha \tag{2.6}$$

zusammengefaßt, der die Komponente

$$\mathfrak{E} = \{cF_{10},\ cF_{20},\ cF_{30}\}, \qquad \mathfrak{B} = \{F_{23},\ F_{31},\ F_{12}\}$$

besitzt. Weiterhin, die Quantitätsgrößen, d.h. der Verschiebungsvektor $\mathfrak{d}$ und die magnetische Feldstärke $\mathfrak{H}$, werden durch den Erreungstensor $G_{\alpha\beta}$ beschrieben, der die Komponente

$$\mathfrak{d} = \left\{\frac{1}{c}\,G_{10},\ \frac{1}{c}\,G_{20},\ \frac{1}{c}\,G_{30}\right\}, \qquad \mathfrak{H} = \{G_{23},\ G_{31},\ G_{12}\} \tag{2.7}$$

hat. Der Zusammenhang zwischen den beiden Tensoren $F_{\alpha\beta}$ und $G_{\alpha\beta}$ ist durch die Materialgleichungen

$$G_{\alpha\beta} = \frac{1}{\mu}\left\{F_{\alpha\beta} + \frac{\varepsilon_{(r)}\mu_{(r)} - 1}{c^2}\left[F_\alpha u_\beta - F_\beta u_\alpha\right]\right\} \tag{2.8}$$

$$\left(\varepsilon_{(r)} = \frac{\varepsilon}{\varepsilon_0},\ \mu_{(r)} = \frac{\mu}{\mu_0},\ F_\alpha = F_{\alpha\lambda}u^\lambda\right),$$

festgestellt, wo $u = \{u^\lambda\}$ die Vierergeschwindigkeit des betrachteten Koordinatensystems relativ zu dem Dielektrikum bedeutet. Im Falle des Ruchsystems des Dielektrikums hat u die Komponenten $\{c, 0, 0, 0\}$.

Die Feldgleichungen

$$\partial_\alpha G^{\lambda\alpha} = -\mu_0\, s^\lambda \tag{2.9}$$

lassen sich von der Lagrangeschen Funktion des Feldes

$$L = \tfrac{1}{4} F_{\alpha\beta}G^{\alpha\beta} - \Phi_\lambda s^\lambda \tag{2.10}$$

mit Hilfe des Variationsprinzips

$$\delta\int L\sqrt{|g|}\,\mathrm{d}^4x = 0\,, \tag{2.11}$$

ableiten.

Es läßt sich unmittelbar zeigen, daß — wenn das Viererpotential der Lorentzschen Bedingung

$$\partial_\alpha \Phi^\alpha = 0 \tag{2.12}$$

Genüge leistet — die Feldgleichungen mit der Wellengleichung (2.4) equivalent sind. Weiterhin, ist es ja bekannt, daß sich auf Grund der Definition des Feldtensors (2.6) die Identitäten

$$\partial_\lambda F_{\alpha\beta} + \partial_\alpha F_{\beta\lambda} + \partial_\beta F_{\lambda\alpha} = 0 \tag{2.13}$$

feststellen lassen.

Im Vakuum reduzieren sich die Materialgleichungen (2.8) auf

$$\underset{(v)}{G}_{\alpha\mu} = \frac{1}{\mu_0} F_{\alpha\beta} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot c F_{\alpha\beta} \,. \tag{2.14}$$

3. – Die Erhaltungsätze.

Das Wirkungsintegral

$$J = \int_\Omega \mathfrak{L}(x)\, \mathrm{d}^4 x \,, \qquad (\mathfrak{L} = \sqrt{|g|}\, L) \tag{3.1}$$

ist eine Integralinvariante, d.h. J beibt bei der Durchführung jeder Koordinatentransformation unverändert:

$$\delta J = 0 \,, \tag{3.2}$$

wo δJ die totale Variation

$$\delta J \stackrel{\text{def}}{=} J'(x') - J(x)$$

bedeutet. Es sei nun die infinitesimale Koordinatentransformation

$$x^{\mu'} = x^\mu + \theta \xi^\mu(x)$$

betrachtet, wo $\xi^\mu(x)$ die kontravarianten Komponenten eines beliebigen Vierervektors bedeuten, welche am Rand des Integrationsgebietes Ω verschwinden sollen und θ einen infinitesimalen Parameter bezeichnet. Durch bekannte Unformungen läßt sich zeigen [4], daß unsere Gleichung (3.2) auch in der Form

$$\int_\Omega \delta^* \mathfrak{L}\, \mathrm{d}^4 x = 0 \,, \tag{3.3}$$

geschrieben werden kann, wo $\delta^*\mathfrak{L}$ die sog. lokale Variation von

$$\delta^*\mathfrak{L} \overset{\text{def}}{=} \mathfrak{L}'(x) - \mathfrak{L}(x)$$

bezeichnet.

Um die allgemeinen Erhaltungsätze ableiten zu können, wollen wir in diesem Praragraphen gemäß dem bekannten Verfahren (4.8) $g^{\varkappa\beta} = g^{\varkappa\beta}(x)$ voraussetzen. In diesem Falle sei aber darauf hingewiesen, daß die Viererstromdichte ihrem Transformationscharakter nach eine Vektorendichte ist, d.h. sie ist in der Gestallt $\mathfrak{s}^\mu/\sqrt{|g|}$ von den Komponenten des metrischen Grundtensors unabhängig. Somit läßt sich die Lagrangesche Funktion des Feldes jetzt auf Grund von (2.10) und (2.8) in der expliziten Form

$$\mathfrak{L} = \sqrt{|g|}\left\{\frac{\mu_0}{4\mu_{(r)}}\left[g^{\alpha\varrho}g^{\beta\sigma}F_{\alpha\beta}F_{\varrho\sigma} + \frac{\varepsilon_{(r)}\mu_{(r)}-1}{c^2}F_{\alpha\beta}(g^{\alpha\varrho}u^\beta - g^{\beta\varrho}u^\alpha)F_{\varrho\lambda}u^\lambda\right] - \frac{\mathfrak{s}^\alpha}{\sqrt{|g|}}\varphi_\alpha\right\},$$

schreiben, d.h.

$$\mathfrak{L} = \mathfrak{L}(g^{\alpha\beta};\ \Phi_\alpha;\ \Phi_{\alpha,\lambda};\ \mathfrak{s}^\alpha;\ \varepsilon_{(r)};\ \mu_{(r)};\ u^\alpha)$$

ist, folglich ergibt sich

$$\delta^*\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial g^{\alpha\beta}}\delta^* g^{\alpha\beta} + \frac{\partial\mathfrak{L}}{\partial\varphi_\alpha}\delta^*\varphi_\alpha + \frac{\partial\mathfrak{L}}{\partial\varphi_{\alpha,\lambda}}\delta^*\varphi_{\alpha,\lambda} +$$

$$+ \frac{\partial\mathfrak{L}}{\partial\mathfrak{s}^\alpha}\delta^*\mathfrak{s}^\alpha + \frac{\partial\mathfrak{L}}{\partial\varepsilon_{(r)}}\delta^*\varepsilon_{(r)} + \frac{\partial\mathfrak{L}}{\partial\mu_{(r)}}\delta^*\mu_{(r)} + \frac{\partial\mathfrak{L}}{\partial u^\alpha}\delta'^* u^\alpha,$$

wo

$$\delta'^* u^\alpha \overset{\text{def}}{=} \delta^* u^\alpha - \frac{\partial u^\alpha}{\partial g^{\alpha\beta}}\delta^* g^{\alpha\beta},$$

bezeichnet. Hier haben wir berücksichtigt, daß bei der Berechnung von $\partial\mathfrak{L}/\partial g^{\alpha\beta}$ schon die Abhängigkeit der Vierergeschwindigkeit von der Komponenten des metrischen Grundtensors im Betracht gezogen war. Nach einer partiellen Integration und durch Ausführen der lokalen Variationen läßt sich unsere Gleichung (3.3) auf die Form

$$\int_\Omega \theta\left\{\partial_\varrho\left[\mathfrak{T}'^{\varrho}_{\lambda\cdot} - \left(\frac{\partial\mathfrak{L}}{\partial u_\lambda} - \frac{\partial\mathfrak{L}}{\partial u^\sigma}u^\sigma u_\lambda\right)u^\varrho\right] + \frac{1}{2}\left[\mathfrak{T}'_{\alpha\beta} + g_{\alpha\beta}\frac{\partial\mathfrak{L}}{\partial\mathfrak{s}^\varrho}\mathfrak{s}^\varrho\right](\partial_\lambda g^{\alpha\beta}) +\right.$$

$$\left. + \left[\partial_\alpha\frac{\partial\mathfrak{L}}{\partial\mathfrak{s}^\lambda} - \partial_\lambda\frac{\partial\mathfrak{L}}{\partial\mathfrak{s}^\alpha}\right]\mathfrak{s}^\alpha - \frac{\partial\mathfrak{L}}{\partial\varepsilon_{(r)}}\partial_\lambda\varepsilon_{(r)} - \frac{\partial\mathfrak{L}}{\partial\mu_{(r)}}\partial_\lambda\mu_{(r)}\right\}\xi^\lambda\, d^4x = 0,$$

bringen, wenn man berücksichtigt, daß

$$\frac{\partial \mathfrak{L}}{\partial \varphi_\alpha} - \partial_\lambda \frac{\partial \mathfrak{L}}{\partial \varphi_{\alpha.\lambda}} = 0\,,$$

mit den Feldgleichungen (2.9) identisch ist und

$$\mathfrak{T}'_{\alpha\beta} \stackrel{\text{def}}{=} -2\,\frac{\partial \mathfrak{L}}{\partial g^{\alpha\beta}}\,, \tag{3.4}$$

bezeichnet. Diese Gleichung soll für beliebige ξ^λ und für die beliebige Wahl des Integrationsgebietes Ω identisch erfüllt werden. Somit bekommen wir die wichtigen Identitäten

$$\partial_\varrho\left\{\mathfrak{T}'^{\varrho}_{\lambda\cdot} - \left(\frac{\partial \mathfrak{L}}{\partial u^\lambda} - \frac{\partial \mathfrak{L}}{\partial u^\sigma}\,u^\sigma u_\lambda\right)u^\varrho\right\} + \frac{1}{2}\left\{\mathfrak{T}'_{\alpha\beta} + g_{\alpha\beta}\,\frac{\partial \mathfrak{L}}{\partial \mathfrak{s}^\varrho}\,\mathfrak{s}^\varrho\right\}(\partial_\lambda g^{\alpha\beta}) = \tag{3.5}$$
$$= \left\{\partial_\lambda \frac{\partial \mathfrak{L}}{\partial \mathfrak{s}^\alpha} - \partial_\alpha \frac{\partial \mathfrak{L}}{\partial \mathfrak{s}^\lambda}\right\}\mathfrak{s}^\alpha + \frac{\partial \mathfrak{L}}{\partial \varepsilon_{(r)}}\,\partial_\lambda \varepsilon_{(r)} + \frac{\partial \mathfrak{L}}{\partial \mu_{(r)}}\,\partial_\lambda \mu_{(r)}\,,$$

oder im pseudo-Euklidischem Raum

$$\partial_\varrho\left\{T''^{\varrho}_{\lambda\cdot} - \left(\frac{\partial L}{\partial u_\lambda} - \frac{\partial L}{\partial u_\sigma}\,u^\sigma u_\lambda\right)u^\varrho\right\} = \left\{\partial_\lambda \frac{\partial L}{\partial s_\alpha} - \partial_\alpha \frac{\partial L}{\partial s_\lambda}\right\}s^\alpha + \frac{\partial L}{\partial \varepsilon_{(r)}}\,\partial_\lambda \varepsilon_{(r)} + \frac{\partial L}{\partial \mu_{(r)}}\,\partial_\lambda \mu_{(r)}\,, \tag{3.6}$$

mit

$$T''^{\varrho}_{\lambda\cdot} \stackrel{\text{def}}{=} (T'^{\varrho}_{\lambda\cdot})_{g^{\alpha\beta} = g^{\alpha\beta}_{\text{ps.-E.}}}\,, \tag{3.7}$$

wo $g^{\alpha\beta}_{\text{ps.-E.}}$ den metrischen Grundtensor des pseudo-Euklidischen Raumes bezeichnet. Diese Identitäten liefern für uns die Erhaltungssätze des elektromagnetischen Feldes.

4. – Diskussionen.

Durch ausführliche, aber elementare Rechnungen erhalten wir auf Grund unserer Lagrangeschen Funktion (2.10) für $T''_{\alpha\beta}$ den bekannten Abrahamschen Energie-Impulstensor des Feldes

$$T''_{\alpha\beta} = -\left\{F_{\alpha\varrho}G_{\beta\cdot}^{\ \varrho} - \frac{1}{4}\,g_{\alpha\beta}F_{\varrho\sigma}G^{\varrho\sigma}\right\} - \left\{\frac{\varepsilon_{(r)}\mu_{(r)} - 1}{\mu c^2}\,[F_{\alpha\varrho}F^{\varrho}u_\beta + F_\varrho F^{\varrho}u_\alpha u_\beta]\right\}, \tag{4.1}$$

bzw. für

$$T_{\alpha\beta} \stackrel{\text{def}}{=} T''_{\alpha\beta} - \left\{\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial u^\sigma}\,u^\sigma u_\alpha\right\}u_\beta\,, \tag{4.2}$$

den Minkowskischen Energie-Impulstensor

$$T_{\alpha\beta} = -\left\{F_{\alpha\varrho} G_{\beta\cdot}^{\varrho} - \tfrac{1}{4} g_{\alpha\beta} F_{\varrho\sigma} G^{\varrho\sigma}\right\}. \tag{4.3}$$

Man sieht unmittelbar ein, daß die beiden Energie-Impulstensoren den Erhaltungssätzen genügen. Die Gleichungen (3.6) lassen sich nämlich entweder in der Form

$$-\partial_\alpha T_{\lambda\cdot}^{\alpha} = F_{\lambda\alpha} s^\alpha - \frac{\partial L}{\partial \varepsilon_{(r)}} \partial_\lambda \varepsilon_{(r)} - \frac{\partial L}{\partial \mu_{(r)}} \partial_\lambda \mu_{(r)}, \tag{4.4}$$

oder in der Form

$$-\partial_\alpha \left\{T_{\lambda\cdot}^{\prime\prime\alpha} - T_{\lambda\cdot}^{*\alpha}\right\} = F_{\lambda\alpha} s^\alpha - \frac{\partial L}{\partial \varepsilon_{(r)}} \partial_\lambda \varepsilon_{(r)} - \frac{\partial L}{\partial \mu_{(r)}} \partial_\lambda \mu_{(r)}, \tag{4.5}$$

schreiben, wo

$$T_{\alpha\beta}^{*} \overset{\text{def}}{=} -\frac{\varepsilon_{(r)}\mu_{(r)} - 1}{\mu c^2}\left\{F_{\alpha\varrho} F^{\varrho} u_\beta + F_\varrho F^\varrho u_\alpha u_\beta\right\}, \tag{4.6}$$

bezeichnet. Das bedeutet aber — im Gegensatz zu den Behauptungen von mehreren Forschern ([1,3,7-9]), die aus der expliziten Form der ponderomotorischen Kräfte auf die exklusive Richtigkeit des Abrahamschen Tensors hingewisen haben —, daß sich allein auf Grund der Erhaltungssätze nicht entscheiden läßt, ob der Abrahamsche, oder der Minkowskische Tensor als richtiger Energie-Impulstensor des Feldes betrachtet werden kann.

Wenn wir aber jetzt uns mit dem Falle des Vakuums beschäftigen wollen, dann sieht man unmittelbar ein, daß der Tensor $T_{\alpha\beta}^{*}$ im Vakuum identisch verschwindet und sich die beiden Energie-Impulstensoren auf

$$\underset{(v)}{T}_{\alpha\beta} = -\left\{F_{\alpha\varrho} \underset{(v)}{G}_{\beta\cdot}^{\varrho} - \tfrac{1}{4} g_{\alpha\beta} F_{\varrho\sigma} \underset{(v)}{G}^{\varrho\sigma}\right\} \tag{4.7}$$

reduzieren. *Die Konstruktion des Tensors* $\underset{(v)}{T}_{\alpha\beta}$ *stimmt aber mit dem Minkowskischen Ansatz überein. Daraus wollen wir den Schluß ziehen, daß der richtige Energie-Impulstensor des elektromagnetischen Feldes nicht nur im Vakuum, sondern auch in Dielektriken der Minkowskische ist.*

Es sei darauf hingewiesen, daß der Tensor $\underset{(v)}{T}_{\alpha\beta}$ in seinen Indizen α und β symmetrisch ist, was man unmittelbar einsehen kann, da er sich auf Grund von der Gleichung (2.14) auch in der Form

$$\underset{(v)}{T}_{\alpha\beta} = -c\sqrt{\frac{\varepsilon_0}{\mu_0}}\left\{F_{\alpha\varrho} F_{\beta\cdot}^{\varrho} - \frac{1}{4} g_{\alpha\beta} F_{\sigma\varrho} F^{\varrho\sigma}\right\},$$

schreiben läßt.

In Dielektriken ist natürlich der Minkowskische Tensor (4.3) nicht mehr symmetrich, obwohl er die Identität (4.4) erfüllt, weswegen die wichtige Plancksche Identität

$$\mathfrak{g} = \frac{1}{c_2}\mathfrak{S} ,$$

(wo $\mathfrak{g}$ die Impulsdichte und $\mathfrak{S}$ den Poyntingschen Vektor das Feldes bezeichnet) nicht besteht. Das hängt aber damit zusammen, daß das elektromagnetische Feld in Dielektriken kein abgeschlossenes System ist, wie darauf schon J. G. TAMM (2) hingewiesen hat. Um das Feld zu einem abgeschlossenen System zu ergänzen, haber wir das sog. Strahlungsfeld einzuführen, dessen Energie-Impulstensor von F. BECK (2) angegeben wurde.

Dadurch war es für uns möglich auf die Richtigkeit des Minkowskischen Ansatzes betreffs des Energie-Impulstensors des Feldes zu schließen, daß wir das Giorgi-Sommerfeldsche System benützt haben. *Die Formulierung der Theorie soll aber dimensionsinvariant sein*, d.h. unser Schluß kann als allgemeingültig bezeichnet werden.

RIASSUNTO (*)

Si dimostra in base del sistema di Giorgi-Sommerfeld che la costruzione del tensore energia-quantità di moto coincide con l'impostazione di Minkowski. Dato che la formulazione della teoria deve essere invariante rispetto alle dimensioni, si conclude che il tensore di Minkowski energia-quantità di moto del campo può essere ritenuto quello giusto.

(*) *Traduzione a cura della Redazione.*

Zur Geometrisierung des elektromagnetischen Feldes.

J. I. Horváth

Institut für Theoretische Physik der Universität - Szeged, Ungarn

(ricevuto il 22 Ottobre 1957)

Zusammenfassung. — Es läßt sich zeigen, daß, durch eine Verallgemeinerung der Levi-Civitaschen Parallelverschiebung, eine Möglichkeit gibt, das Vorhandensein des elektromagnetischen Feldes in einem angegebenen Gravitationsfeld im dem vier-dimensionalen Riemannschen Raum rein geometrisch zu interpretieren.

Die feldtheoretischen Untersuchungen, die seit der begeisternden Faradayschen Idee eine der wichtigsten und brauchbarsten Anschauungsformen der modernen Physik geliefert haben, sind in den drei letzten Jahrzehnten in eine entscheidende Entwicklungsperiode getreten. Est ist nämlich wohlbekannt, daß die Feldtheorie im engeren Sinne (*) — seit der Einsteinschen allgemeinen Relativitätstheorie — nach dem Ziele strebte, alle physikalischen Wirkungen, deren Feldtheorien im weiteren Sinne schon ausgearbeitet sind, durch die geometrische Struktur der vierdimensionalen Raum-Zeitmannigfaltigkeit, in welcher sich die Naturerscheinungen abspielen, darzustellen. Diese Untersuchungen, die man in das Gebiet der einheitlichen Feldtheorie einzureihen pflegt, sind auch heute noch nicht in zufriedenstellender Weise abgeschlossen. Die Schwierigkeiten, die die Forschung im Falle der Gravitation und des Elektromagnetismus schon seit mehr als 30 Jahren verschiedentlich vergebens zu

(*) Ist die Feldtheorie in einem pseudo-euklidischen Raum entwickelt, dann wollen wir die Theorie als *Feldtheorie im weiteren Sinne* bezeichnen (z.B. die Maxwell-Lorentzsche Theorie, oder die Theorie der Mesonfelder). Hingegen, benennen wir als *Feldtheorie im engeren Sinne* diejenigen Feldtheorien, die der ursprünglichen Riemannschen Idee gemäß die physikalischen Wirkungen des Führungsfeldes unmittelbar durch die geometrische Struktur des Raumes darstellen. (Z.B. die allgemeine Relativitätstheorie).

bewältigen suchte, können wir damit erklären, daß die Geometrisierung des elektromagnetischen Feldes noch nicht in zufriedenstellender Weise gelang. Die Untersuchungen, die sich mit diesem Problem beschäftigt haben, gingen von der allgemein akzeptierten Festsetzung aus, daß der vierdimensionale Riemannsche Raum, welcher der allgemeinen Relativitätstheorie zugrunde liegt, für eine derartige Interpretation jedenfalls keine geeignete Größe hat. Wenn man nämlich die Krümmung des Raumes für das Gravitationsfeld beansprucht, hat die Riemannsche Geometrie kein geometrisches Objekt, das für die Darstellung der elektromagnetische Wirkung durch ein elektromagnetisches Füghrungsfeld geeignet wäre. Um das Grundproblem der einheitlichen Feldtheorie lösen zu können, schien es also notwendig zu sein, das Weltkontinuum als eine Mannigfaltigkeit von allgemeinerer Struktur aufzufassen.

Wir sind noch nicht in der Lage, daß wir im nachfolgenden eine endgültige Geometrisierung für das elektromagnetische Feld und damit eine vollständige einheitliche Feldtheorie für Gravitation und Elektromagnetismus angeben können. Doch, wie wir es in dieser Untersuchung noch eingehender ausführen wollen, läßt sich zeigen, daß es auch in der vierdimensionalen Riemannschen Geometrie eine Möglichkeit gibt das Vorhandensein des elektromagnetischen Feldes in einem angegebenen Gravitationsfeld rein geometrisch zu verstehen. Die Grundlage für diese Möglichkeit bildet die Tatsache, daß sich die metrische Parallelübertragung, wie darauf schon P. Nalli (1), J. A. Schouten (2) und neuerdings im Falle der allgemeinen Linienelementräume A. Moór (3) hingewiesen haben, auch in der Riemannschen Geometrie noch etwas verallgemeinern läßt.

1. – Geometrische Vorbereitungen.

Es sein ein gewöhnlicher Riemannscher Raum angegeben, dessen Metrik durch den metrischen Grundtensor $g_{ik} = g_{ik}(x)$ festgelegt ist. In diesem Raum ist die Parallelübertragung in gewöhnlicher Weise mit Hilfe der symmetrischen Christoffelschen Symbole

$$\Gamma_{ijk} = \tfrac{1}{2}\{\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik}\}$$

in der Form

$$D\xi^i = \mathrm{d}\xi^i + \Gamma_{r\cdot s}^{\;i}\xi^r\,\mathrm{d}x^s = 0$$

(1) P. Nalli: *Rend. Acc. Linc.*, (6), **13**, 669 (1931).
(2) J. A. Schouten: *Ricci-Calculus* (Berlin, 1954).
(3) A. Moór: *Acta Sci. Math. Szeged*, **17**, 85 (1956).

gegeben. Diese Übertragung ist eine metrische Übertragung, d.h. es ist

$$Dg^{ik} = 0 .$$

Es läßt sich aber unmittelbar einsehen, daß in dem Riemannschen Raum eine allgemeinere metrische Übertragung definiert werden kann. Sei nämlich die metrische Übertragung durch

$$D^*\xi^i \overset{\text{def}}{=} \mathrm{d}\xi^i + L_{r\cdot s}^{\ i}\,\xi^r\,\mathrm{d}x^s \tag{1.1}$$

festgelegt, wo die

$$L_{rjs} \overset{\text{def}}{=} \Gamma_{rjs} + \Lambda_{rjs} \tag{1.2}$$

sind, dann ist die notwendige und hinreichende Bedingung dafür, daß die Übertragung D^* metrisch sei, die Voraussetzung, daß Λ_{rjs} einen in seinen beiden ersten Indices r und j schiefsymmetrischen Tensor bedeuten soll

$$\Lambda_{rjs} = -\Lambda_{jrs} . \tag{1.3}$$

Damit erhalten wir eine natürliche Verallgemeinerung des Riemannschen Raumes, welcher dieselbe Metrik hat, wie der ursprüngliche Raum. Doch ist der Krümmungstensor dieser allgemeineren Riemannschen Mannigfalltigkeit durch

$$T_{i\cdot km}^{\ j} \overset{\text{def}}{=} R_{i\cdot km}^{\ j} + P_{i\cdot km}^{\ j} \tag{1.4}$$

gegeben, wo

$$R_{i\cdot km}^{\ j} \overset{\text{def}}{=} \partial_m \Gamma_{i\cdot k}^{\ j} - \partial_k \Gamma_{i\cdot m}^{\ j} + \Gamma_{i\cdot k}^{\ r}\Gamma_{r\cdot m}^{\ j} - \Gamma_{i\cdot m}^{\ r}\Gamma_{r\cdot k}^{\ j} \tag{1.5}$$

den Krümmungstensor des ursprünglichen Riemannschen Raumes und

$$P_{i\cdot km}^{\ j} \overset{\text{def}}{=} \Lambda_{i\cdot k|m}^{\ j} - \Lambda_{i\cdot m|k}^{\ j} + \Lambda_{i\cdot k}^{\ r}\Lambda_{r\cdot m}^{\ j} - \Lambda_{i\cdot m}^{\ r}\Lambda_{r\cdot k}^{\ j} \tag{1.6}$$

bedeuten. Dabei bedeutet die Operation « $|m$ » die kovariante Ableitung bezüglich der Übertragungsparameter $\Gamma_{i\cdot k}^{\ j}$.

Die Definition (1.4) des Krümmungstensors ist gleichbedeutend mit

$$T_{i\cdot km}^{\ j} = \partial_m L_{i\cdot k}^{\ j} - \partial_k L_{i\cdot m}^{\ j} + L_{i\cdot k}^{\ r} L_{r\cdot m}^{\ j} - L_{i\cdot m}^{\ r} L_{r\cdot k}^{\ j} . \tag{1.7}$$

Unsere beiden Tensoren (1.4) und (1.6) haben die folgenden Symmetrieeigenschaften

$$T_{ijkm} = -T_{jikm}, \quad T_{ijkm} = -T_{ijmk}, \quad P_{ijkm} = -P_{jikm}, \quad P_{ijkm} = -T_{ijmk} . \tag{1.8}$$

Bezeichnet man die kovariante Ableitung des Vektors ξ^i bezügliche der Übertragungsparameter $L_{i\cdot k}^{\cdot j}$ mit $\xi^{i}_{\cdot;k}$, so für einen Vektor

(1.9) $$\xi^{i}_{\cdot;k} = \xi^{i}_{\cdot|k} + \Lambda_{r\cdot k}^{\cdot i}\xi^r, \qquad \text{bzw.} \qquad \xi_{i;k} = \xi_{i|k} + \Lambda_{i\cdot k}^{\cdot r}\xi_r .$$

Die wohlbekannten Identitäten von Ricci sind dann für einen gemischten Tensor $A_{i\cdot}^{\cdot j}$:

(1.10) $$A_{i\cdot;km}^{\cdot j} - A_{i\cdot;mk}^{\cdot j} = T_{r\cdot km}^{\cdot j} A_{i\cdot}^{\cdot r} - T_{i\cdot km}^{\cdot r} A_{r\cdot}^{\cdot j} - 2A_{i\cdot|r}^{\cdot j} Q_{k\cdot m}^{\cdot r},$$

wo

(1.11) $$Q_{k\cdot m}^{\cdot r} \overset{\text{def}}{=} \tfrac{1}{2}\left\{L_{k\cdot m}^{\cdot r} - L_{m\cdot k}^{\cdot r}\right\} = \tfrac{1}{2}\left\{\Lambda_{k\cdot m}^{\cdot r} - \Lambda_{m\cdot k}^{\cdot r}\right\}$$

bedeutet. $Q_{k\cdot m}^{\cdot r}$ ist also der schiefsymmetrische Anteil der allgemeinen metrischen Übertragung $L_{k\cdot m}^{\cdot r}$; doch wollen wir noch darauf hinweisen, daß $L_{k\cdot m}^{\cdot r}$ noch nicht den symmetrischen Anteil von $L_{k\cdot m}^{\cdot r}$ bildet. Nach (1.3) ist nämlich

$$L_{(k\cdot m)}^{\cdot r} \overset{\text{def}}{=} \tfrac{1}{2}\left\{L_{k\cdot m}^{\cdot r} + L_{m\cdot k}^{\cdot r}\right\} = \Gamma_{k\cdot m}^{\cdot r} + \tfrac{1}{2}\left\{\Lambda_{k\cdot m}^{\cdot k} + \Lambda_{m\cdot k}^{\cdot r}\right\}.$$

Nur wenn $\Lambda_{k\cdot m}^{\cdot r}$ auch in seinen Indices k und m schiefsymmetrisch ist, bestimmt $\Gamma_{k\cdot m}^{\cdot r}$ den symmetrischen Teil von $L_{k\cdot m}^{\cdot r}$. Dieser Spezialfall hat aber vom Standpunkt der Feldtheorie aus keine Bedeutung.

Nach (1.9), (1.4) und (1.7) erhält man aus (1.10)

(1.12) $$A_{i\cdot|km}^{\cdot i} - A_{i\cdot|mk}^{\cdot j} = R_{r\cdot km}^{\cdot j} A_{i\cdot}^{\cdot r} - R_{i\cdot km}^{\cdot r} A_{r\cdot}^{\cdot j} .$$

Unsere Identität (1.10) geht also für die kovarianten Ableitungen bezüglich der Übertragungsparameter $\Gamma_{i\cdot k}^{\cdot j}$ in die klassischen Identitäten von Ricci über.

Jetzt wollen wir noch die Verallgemeinerung der wichtigen Identitäten von Bianchi angeben. Es läßt sich unmittelbar zeigen, daß die Relationen

(1.13) $$T_{i\cdot\{km|r\}}^{\cdot j} = P_{i\cdot\{km|r\}}^{\cdot j}$$

mit

$$A_{\{ijk\}} \overset{\text{def}}{=} A_{ijk} + A_{jki} + A_{kij} \qquad \text{und} \qquad A_{\{km:t:r\}} \overset{\text{def}}{=} A_{kmtr} + A_{mrtk} + A_{rktm},$$

bzw.

(1.14) $$T_{i\cdot\{km;r\}}^{\cdot j} + T_{i\cdot s\{m}^{\cdot j}\Lambda_{k\cdot r\}}^{\cdot s} = 0,$$

oder durch Einführung von $Q_{k\cdot m}^{\cdot r}$:

(1.15) $$T_{i\cdot\{km;r\}}^{\cdot j} + 2T_{i\cdot\{k:t:}^{\cdot j} Q_{m\cdot r\}}^{\cdot t} = 0$$

identisch erfüllt sind.

Die Gleichungen (1.13)–(1.15) sind alle mit den klassischen Bianchischen Identitäten äquivalent. Sie sind nämlich auch in $\Lambda_{i\cdot k}^{\;j}$ Identitäten, somit, wenn man (1.7), (1.9), bzw. (1.11) beachtet, fallen alle Glieder, die die $\Lambda_{i\cdot k}^{\;j}$ enthalten, aus und man erhält die klassischen Identitäten

$$R_{i\cdot\{km|r\}}^{\;j} = 0\,. \tag{1.16}$$

Aus (1.13) folgt nach einer Überschiebung mit $g^{ik}\delta_j^{\;m}$ die Relation

$$\left\{T^{i}_{\cdot r} - P^{i}_{\cdot r} - \tfrac{1}{2}\delta^{i}_{\;r}(T-P)\right\}_{|i} = 0\,, \tag{1.17}$$

wo

$$T_{ir} \stackrel{\text{def}}{=} T_{i\cdot rs}^{\;s}$$

die Verallgemeinerung des Einstein-Riccischen Tensors und

$$T \stackrel{\text{def}}{=} g^{ir}\,T_{ir} \tag{1.18}$$

den verallgemeinerten Krümmungsskalar bedeuten. Entsprechenderweise sind die Defiinition von P_{ir} und P durch

$$P_{ir} \stackrel{\text{def}}{=} P_{i\cdot rs}^{\;s}\,, \qquad \text{bzw.} \qquad P \stackrel{\text{def}}{=} g^{ir}P_{ir} \tag{1.19}$$

angegeben.

Aus (1.14) läßt sich nach einer Überschiebung mit $g^{ik}\delta_j^{\;m}$ die folgende wichtige Identität

$$\left\{T^{i}_{\cdot r} - \tfrac{1}{2}\delta^{i}_{r}T\right\}_{;i} + 2T^{i}_{\cdot s}Q_{r\cdot i}^{\;s} + T^{ij}_{\cdot\cdot rs}Q_{i\cdot i}^{\;s} = 0\,, \tag{1.20}$$

bzw. aus (1.15) durch eine Überschiebung mit $g^{ik}\delta_j^{\;m}$ die Identität

$$\left\{T^{i}_{\cdot r} - \tfrac{1}{2}\delta^{i}_{\cdot r}T\right\}_{;i} + \tfrac{1}{2}\left\{T^{jis}_{\cdot\cdot\cdot r} + \delta^{i}_{\;r}T^{js}\right\}Q_{jsi} = 0 \tag{1.21}$$

ableiten.

Die angegebene Verallgemeinerung der Riemannschen Geometrie befindet sich im wesentlichen in der Tatsache, daß die von dem Variationsprinzip

$$\delta\int\left\{g_{ik}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}\right\}^{\frac{1}{2}}\mathrm{d}s = 0$$

ableitbaren geodetischen Linien des Raumes, deren Gleichungen sich in der wohlbekannten Form

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} + \Gamma_{r\cdot s}^{\;i}\frac{\mathrm{d}x^r}{\mathrm{d}s}\frac{\mathrm{d}x^s}{\mathrm{d}s} = 0 \tag{1.22}$$

angeben lassen, von den autoparallelen Kurven, oder — wie wir sie im folgenden bezeichnen wollen — von den Bahnen des Raumes, die sich durch die Gleichungen

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2}+L_{r\cdot s}^{\,i}\frac{\mathrm{d}x^r}{\mathrm{d}s}\,\frac{\mathrm{d}x^s}{\mathrm{d}s}=0 \tag{1.23}$$

bestimmen lassen, verschieden sind. Hier und im folgenden soll s als Bogenparameter gelten.

2. – Die Grundidee der vorgeschlagenen Theorie.

Befindet sich ein geladener Massenpunkt unter der Einwirkung eines Führungsfeldes in irgendwelchen Bewegungszustand, so ist seine Bahn in dem vierdimensionalen Raum-Zeitkontinuum durch seine Weltlinie angegeben. Wollen wir gemäß der allgemeinen Feldtheorie im engeren Sinne das Führungsfeld durch die geometrische Struktur des Raumes zum Ausdruck bringen, so läßt sich die ponderomotorische Kraft des Führungsfeldes austransformieren und unser Massenpunkt führt eine Inertialbewegung aus, d.h. er bewegt sich längs einer autoparallelen Kurve.

Wir wollen jetzt voraussetzen, daß

$$\Lambda_{ijk}\overset{\mathrm{def}}{=}F_{ij}s_k \tag{2.1}$$

ist, wo F_{ij} den schiefsymmetrischen Feldtensor des elektromagnetischen Feldes und s_k die Viererstromdichte bedeutet. Lokalisiert sich die Viererstromdichte auf die Weltlinie $x^i=x^i(s)$ eines geladenen Massenpunktes, der eine Ladung e besitzt, dann kann man den Viererstrom auf folgende Weise definieren

$$s^k=e\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\,.$$

Mit Hilfe dieser Definition erhält man, daß

$$-\Lambda_{j\cdot k}^{\,i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}=-F_{j\cdot}^{\,i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\,s_k\,\frac{\mathrm{d}x^k}{\mathrm{d}s}=-eF_{j\cdot}^{\,i}\frac{\mathrm{d}x^j}{\mathrm{d}s}=F^{i}_{\cdot j}s^j=f^i$$

die ponderomotorische Lorentz-Kraft darstellt. Wir haben dabei die Relation

$$g_{rs}\frac{\mathrm{d}x^r}{\mathrm{d}s}\,\frac{\mathrm{d}x^s}{\mathrm{d}s}=1$$

benützt. Somit läßt sich die Gleichung (1.23) der autoparallelen Kurven in der Form

$$\frac{d^2x^i}{ds^2} + \Gamma_{j\cdot k}^{\;i}\frac{dx^j}{ds}\frac{dx^k}{ds} = f^i \tag{2.2}$$

schreiben, und wir haben die richtige Bewegungsgleichung des geladenen Massenpunktes erhalten.

Definieren wir die Viererstromdichte durch

$$s^k = \varrho_0(x)\frac{dx^k}{ds},$$

wo $\varrho_0(x)$ die skalare Ruheladungsdichte bedeutet, dann ist

$$-\Lambda_{j\cdot k}^{\;i}\frac{dx^j}{ds}\frac{dx^k}{ds} = -\varrho_0 F_{j\cdot}^{\;i}\frac{dx^j}{ds} = k^i$$

die ponderomotorische Lorentz-Kraftdichte und unsere Bewegungsgleichung lautet:

$$\frac{d^2x^i}{ds^2} + \Gamma_{j\cdot k}^{\;i}\frac{dx^j}{ds}\frac{dx^k}{ds} = k^i\,. \tag{2.3}$$

Es ergibt sich somit, daß *die Lorentzsche Kraft unserem heuristischen Ansatz gemäß unmittelbar die Abweichung der autoparallelen Kurven von den Geodäten festlegt.* Besitzt das Kontinuum, oder der Massenpunkt keine Ladung, bzw. ist kein elektromagnetisches Führungsfeld vorhanden, dann verschwindet entweder F_{ij}, oder s_k, d.h. es fallen die autoparallelen Kurven mit den geodetischen Linien des Raumes zusammen und wir bleiben demnach im Rahmen der Einsteinschen Gravitationstheorie. Damit ist also die einfachste Geometrisierung des elektromagnetischen Feldes gelungen.

Jetzt wollen wir noch einige wichtige Formeln unserer Theorie ableiten. Zuerst haben wir den Tensor $P_{i\cdot km}^{\;j}$ auf Grund unseres Ansatzes (2.1) explizit zu berechnen:

$$P_{i\cdot km}^{\;j} = F_{i\cdot|m}^{\;j}s_k - F_{i\cdot|k}^{\;j}s_m + F_{i\cdot}^{\;j}(s_{k|m} - s_{m|k}) \tag{2.4}$$

und — nach (1.19) —

$$P_{ik} = F_{i\cdot|j}^{\;j}s_k - F_{i\cdot|k}^{\;j}s_j + F_{i\cdot}^{\;j}(s_{k|j} - s_{j|k})\,. \tag{2.5}$$

In der gewöhnlichen Riemannschen Geometrie spielt der Krümmungsskalar

$$R \overset{\text{def}}{=} g^{ik}R_{i\cdot ks}^{\;s} \tag{2.6}$$

eine entscheidende Rolle. Die entsprechende Größe ist jetzt in unserem Falle

$$T = R + P\,, \tag{2.7}$$

wo

$$P = 2(F^{ij}s_i)_{|j} = -2k^{j}_{\cdot|j} \tag{2.8}$$

bedeutet.

3. – Deduktiver Aufbau der angegebenen Theorie.

Die Feldgleichungen der angegebenen Theorie werden wir jetzt von einem Variationsprinzip ableiten. Diese Methode selbst, und auch die Rechnung in ihren Einzelheiten, ist von der früheren Theorie her wohlbekannt [1], daher wollen wir sie kurz skizzieren und nur die Lagrangesche Funktion angeben.

Als Lagrangesche Funktion wollen wir jetzt die gewöhnliche der allgemeinen Relativitätstheorie im materieerfüllten Raum betrachten:

$$G = R + 2\varkappa L + 2\varkappa M\,, \tag{3.1}$$

wo R den Riemannschen Krümmungsskalar (2.6), L die Lagrangesche Funktion des elektromagnetischen Feldes

$$L = \tfrac{1}{4}F_{ik}F^{ik} - \Phi_i s^i \tag{3.2}$$

mit

$$F_{ik} = \partial_i \Phi_k - \partial_k \Phi_i \tag{3.3}$$

und M die Lagrangesche Funktion des materiellen Mediums

$$M = -g^{ik}M_{ik} \tag{3.4}$$

bedeuten. Hier sind Φ_i die Komponenten des Viererpotentials und M_{ik} der Energie-Impulstensor unseres Mediums. $\varkappa$ bedeutet darin einen Proportionalitätsfaktor, der von der Wahl der Einheiten abhängt.

Die Feldgleichungen haben wir dann aus dem Hamiltonschen Variationsprinzip

$$\delta\int G\sqrt{|g|}\,\mathrm{d}^4x = 0$$

[4] G. Beck: *Allgemeine Relativitätstheorie* (H. Geiger und K. Schell: *Handbuch der Physik*, Bd. **4**, Berlin, 1929)).

abzuleiten. Bei der Variation wollen wir berücksichtigen, daß die 10 Potentiale des Gravitationsfeldes und die 4 Potentiale des elektromagnetischen Feldes voneinander unabhängig sind, und daß ihre Variationen an den Integrationsgrenzen verschwinden sollen. Es läßt sich unmittelbar zeigen (4), daß die resultierenden Feldgleichungen

$$\sqrt{|g|}\left\{R_{ik}-\tfrac{1}{2}g_{ik}R\right\}=\varkappa\sqrt{|g|}\left\{S_{ik}+M_{ik}\right\}, \tag{3.5}$$

$$\frac{1}{\sqrt{|g|}}\,\partial_k(\sqrt{|g|}\,F^{ik})=s^i \tag{3.6}$$

sind, wo S_{ik} den Energie-Impulstensor des elektromagnetischen Feldes angibt:

$$S_{ik}=-\left\{F_{ir}F_{k\cdot}^{\cdot r}-\tfrac{1}{4}g_{ik}F_{rs}F^{rs}\right\}. \tag{3.7}$$

Die Gleichungen (3.6) bilden zusammen mit (3.3) und mit den kovarianten Identitäten

$$\partial_{\{l}F_{ik\}}=0 \tag{3.8}$$

die kovariante Verallgemeinerung der Maxwell-Lorentzschen Feldgleichungen der Elektrodynamik in Gravitationsfeld.

Zur Vereinfachung des sprachlichen Ausdruckes, wollen wir vorübergehend neben dem gewöhnlichen Riemannschen Raum den verallgemeinerten Raum mit den Übertragungsparametern $L_{i\cdot k}^{\cdot i}$ als *quasi-Riemannschen Raum* bezeichnen.

Wohlbekannterweise ist die Krümmung des Riemannschen Kontinuums durch die Feldgleichungen (3.5) schon festgelegt. Jetzt wollen wir zeigen, daß *die Krümmung des Raumes durch das Vorhandensein des elektromagnetischen Feldes nicht beeinflußt wird.* Durch eine Kontraktion mit g^{ik} erhalten wir nämlich von (3.5), wenn wir den Faktor $\sqrt{|g|}$ zu beiden Seiten der Gleichung weglassen, wegen

$$g^{ik}S_{ik}=-\left\{g^{ik}F_{ir}F_{k\cdot}^{\cdot r}-F_{rs}F^{rs}\right\}\equiv 0$$

die Beziehung

$$R=\varkappa M\,,$$

w.z.b.w.

Das bedeutet aber offensichtlich nicht, daß auch die Krümmung unseres quasi-Riemannschen Raumes dieselbe ist.

Um auch die Krümmung des quasi-Riemannschen Raumes berechnen zu können wollen wir erst darauf hinweisen, daß unsere Gleichung (3.6) auch in der Form

$$F^{ik}_{\cdot\cdot|k}=s^i$$

geschrieben werden kann und auf Grund von (1.9) und (2.1) beweist man leicht, daß auch

$$F^{ik}_{\cdot\cdot;k} = s^i$$

gültig ist.

Aus der Gleichung (2.7) sieht man, daß sich die Krümmung T unseres quasi-Riemannschen Raumes aus der Krümmung R des Riemannschen Raumes und aus dem Skalar P additiv zusammensetzen läßt. Ferner, haben wir in (2.8) gesehen, daß $-P$ von der Divergenz der Lorentzschen Kraftdichte

$$k^i = F^{ij} s_j$$

abhängt. Um unsere Gleichung (2.8) etwas umformen zu können, wollen wir $S_{i\cdot;k}^{\;k}$ berechnen. Wegen der Symmetrie von S_{rk} in k und r wird nach (2.1):

$$S_{i\cdot;k}^{\;k} = S_{i\cdot|k}^{\;k} + \Lambda_{r\cdot k}^{\;k} S_{i\cdot}^{\;r} - \Lambda_{k\cdot i}^{\;r} S_{r\cdot}^{\;k} = S_{i\cdot|k}^{\;k} + F_{r\cdot}^{\;k} s_k S_{i\cdot}^{\;r} - F_{i\cdot}^{\;r} s_k S_{r\cdot}^{\;k}.$$

Auf Grund des wohlbekannten Zusammenhanges

$$\Gamma_{k\cdot r}^{\;r} = \frac{1}{\sqrt{|g|}}\,\partial_k \sqrt{|g|}$$

ergibt sich

$$S_{i\cdot;k}^{\;k} = \frac{1}{\sqrt{|g|}}\,\partial_k(\sqrt{|g|}\,S_{i\cdot}^{\;k}) - \Gamma_{k\cdot i}^{\;r} S_{r\cdot}^{\;k} + F_{r\cdot}^{\;k} s_k S_{i\cdot}^{\;r} - F_{i\cdot}^{\;r} s_k S_{r\cdot}^{\;k}.$$

In der allgemeinen Relativitätstheorie pflegt man zu beweisen [1], daß

$$\partial_k(\sqrt{|g|}\,S_i^{\;k}) - \Gamma_{i\cdot k}^{\;r}\sqrt{|g|}\,S_{r\cdot}^{\;k} = -F_{i\cdot}^{\;k}\sqrt{|g|}\,s_k$$

gilt und damit erhalten wir:

$$S_{i\cdot;k}^{\;k} = F_{r\cdot}^{\;k} s_k S_{i\cdot}^{\;r} - F_{i\cdot}^{\;k} s_k - F_{i\cdot}^{\;r} s_k S_{r\cdot}^{\;k}.$$

Somit ergibt sich

$$F^{ik} s_k = S^{ik}_{\cdot\cdot;k} - F_{r\cdot}^{\;k} s_k S^{ir} - F_{i\cdot}^{\;r} s_k S_{r\cdot}^{\;k} \equiv S_{i\cdot|k}^{\;k}$$

und nach der Definitionsgleichung (2.8) wird

$$P = 2 S^{ik}_{\cdot\cdot|ik}. \tag{3.9}$$

Es ist noch von Interesse P auch ausführlich zu berechnen. Auf Grund der Definition (3.7) von S^{ik} und der Riccischen Identitäten (1.12) bekommt man nach elementarer Rechnung

$$(3.10) \qquad P = 4\{F^{k}_{\cdot r} s^{r}_{\cdot|k} - s_k s^k\} + 2\{F^{kr} F^{is} R_{srik} + F^{k}_{\cdot r} F^{sr} R_{ks}\} + \\ + g^{ik}\{F_{rs} F^{rs}_{\cdot\cdot|ik} + F_{rs|i} F^{rs}_{\cdot\cdot|k}\}.$$

Aus den beiden letzten Gleichungen und aus (2.6) sieht man unmittelbar, daß sich die zusätzliche Krümmung des quasi-Riemannschen Raumes auf das Vorhandensein des elektromagnetischen Feldes zurückführen läßt. Doch zeigt unsere Gleichung (3.10) sehr ausdrücklich, daß diese Wirkung des elektromagnetischen Feldes auch durch das Gravitationsfeld beeinflußt wird.

Bekantlich pflegt man in der allgemeinen Relativitätstheorie zu beweisen, daß sich von den Feldgleichungen (3.5) und (3.6) die richtigen Bewegungsgleichungen (2.3) ableiten lassen. Diese Gleichungen fallen aber, wie wir darauf hingewiesen haben, mit den Gleichungen der autoparallelen Kurven des quasi-Riemannschen Raumes zusammen, was bedeutet, daß ein geladener Massenpunkt, unseren heuristischen Voraussetzungen gemäß, längs einer autoparallelen Kurve des quasi-Riemannschen Raumes eine Inertialbewegung fortführt.

4. – Diskussionen.

1) Erst wollen wir darauf hinweisen, daß es gemäß der in der allgemeinen Relativitätstheorie üblichen Methode natürlich wäre bei der Herleitung der Feldgleichungen aus dem Variationsprinzip

$$\delta \int T \sqrt{|g|}\, \mathrm{d}^4 x = 0$$

auszugehen. Man kann aber leicht nachprüfen, daß dieser Ansatz nicht zu den richtigen Feldgleichungen führt.

2) Man sieht unmittelbar ein, daß unsere Geometrisierung des elektromagnetischen Feldes die bekannten und nachprüfbaren Resultate der allgemeinen Relativitätstheorie nicht beeinflußt. Wenigstens, wenn wir voraussetzen, daß das Photon und auch Merkur keine elektrische Ladung besitzen, die Bahn des Photons mit den geodetischen Linien (1.22) des Raumes zusammenfällt und die Bahn Merkurs läßt sich, wie in der allgemeinen Relativitätstheorie, aus der Gleichung (1.22) berechnen. Die Rotverschiebung ist durch den metrischen Grundtensor g_{ik} festgelegt, d.h. wird durch unsere Geometrisierung des elektromagnetischen Feldes nicht beeinflußt.

3) Man könnte gegen unsere Auffassung darauf hinweisen, daß in einem reinen Strahlungsfelde, oder in Gebieten des Raumes, wo die Viererstromdichte verschwindet, die geometrische Struktur des Raumes durch das elektromagnetische Feld nicht beeinflußt wird; d.h. in Abwesenheit des Gravitationsfeldes die pseudo-Euklidische Struktur des Raumes sich durch das elektromagnetische Vakuum nicht ändern läßt. Wir werden aber jetzt zeigen, daß dieser Einwand sich auf Grund unserer Feldgleichungen vermeiden läßt.

Es ist bekannt, daß unsere Feldgleichungen (3.5) und (3.6) im Falle des elektromagnetischen Vakuums in der Form

$$\sqrt{|g|}\,\{R_{ik}-\tfrac{1}{2}g_{ik}R\}=\varkappa\sqrt{|g|}\,\{S_{ik}+M_{ik}\},$$

bzw.

$$\frac{1}{\sqrt{|g|}}\,\partial_k(\sqrt{|g|}\,F^{ik})=0$$

geschrieben werden können.

Man sieht von diesen Gleichungen unmittelbar ein, daß der Krümmungsskalar R des ursprünglichen Riemannschen Raumes durch das elektromagnetische Feld nicht beeinflußt wird. Das bedeutet jedoch wegen des Auftretens des elektromagnetischen Energie-Impulstensors in den Feldgleichungen nicht, daß die Struktur des ursprünglichen Riemannschen Raumes nicht beeinflußt wird. Weiterhin läßt sich auf Grund von (3.10) leicht einsehen, daß auch der zusätzliche Krümmungsskalar P des quasi-Riemannschen Raumes von Null verschieden ausfällt.

Unser Ansatz hat aber die Folge, daß in diesem Falle die geodetischen Linien und die Bahnen des Raumes zusammenfallen. Das entspricht aber der Tatsache, daß die ponderomotorische Kraftdichte des elektromagnetischen Feldes verschwindet.

4) Unsere Theorie läßt sich auch in einem anisotropen Raum ausbauen. Dann werden die Komponenten des metrischen Grundtensors nicht nur von den Ortskoordinaten, sondern auch von der Richtung abhängig. Mit diesem Problem wollen wir uns in einer nachfolgenden Arbeit beschäftigen.

Zusatz bei dei Korrektur.

Diese Arbeit bildet einen Teil einer nicht-publizierten Arbeit des Verfassers, die noch im Jahre 1955 an die Ungarische Akademie der Wissenschaften eingeschickt wurde. Neuerdings beschäftigte sich L. P. EISENHART (*Proc. Nat. Acad. Sci.*, **42**, 249 (1956))

mit einer sehr ähnlichen Darstellung der Geometrisierung des elektrodynamischen Feldes. Diese Arbeit hat aber der Verfasser nur nach der Einsendung seines Manuskriptes kennengelernt.

RIASSUNTO (*)

Per mezzo di una generalizzazione dello spostamento parallelo di Levi-Civita, si può dimostrare che esiste la possibilità di una interpretazione puramente geometrica dell'esistenza del campo elettromagnetico in un dato campo gravitazionale nello spazio riemanniano a 4 dimensioni.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Mass Reversal and weak Interactions (*).

J. J. Sakurai (+)

Laboratory of Nuclear Studies, Cornell University - Ithaca, N.Y.

(ricevuto il 31 Ottobre 1957)

Summary. — An attempt is made to construct a theory of all weak processes from a single invariance principle. The requirement that the weak interaction Hamiltonian be invariant under the reversal of the sign of the mass in the Dirac equation separately for *any* fermion field leads to a unique form of the four-fermion interaction responsible for weak processes. The Hamiltonian contains equal amounts of V and A, and is necessarily invariant under time reversal but non-invariant under parity regardless of whether or not the neutrino field is involved. (Our results are identical to those obtained by Feynman and Gell-Mann and by Sudarshan and Marshak using somewhat different approaches.) The possible existence of an intermediate *charged* boson field responsible for all weak processes is discussed with reference to several weak processes which have not been observed. Most experiments in β decay and the π-μ-e sequence are consistent with the predictions of our theory with the important exception of the ^{6}He recoil experiment. Various implications of our theory which can be experimentally tested are examined with special attention to the decay interactions of hyperons and K-mesons.

One of the earliest attempts to « explain » the non-conservation of parity was to attribute to the *free* neutrino field a peculiar reflection prperty [1-3]. It has been contended that the very existence of the two-component neutrino

(*) Supported in part by the joint program of the U.S. Atomic Energy Commission and the Office of Naval Research.

(+) Cornell University Senior Graduate Fellow.

(1) A. Salam: *Nuovo Cimento*, **5**, 299 (1957).

(2) L. Landau: *Nuclear Physics*, **3**, 127 (1957).

(3) T. D. Lee and C. N. Yang: *Phys. Rev.*, **106**, 1671 (1957).

guarantees the break-down of the reflection principle so that we have a « natural » understanding of parity non-conservation at least for processes involving neutrinos. However, an attempt to extend this view to the θ and τ-decay modes where the neutrino plays no role has not been found to be successful [4,5]. Moreover, closer examination of the situation seems to reveal that parity non-conservation arises from a particular interaction Hamiltonian itself and not from any peculiar property of the zero-mass fermion, the transformation properties of the *free* neutrino field being more or less a matter of representation [6-8]. Aside from this question, even if we assume the validity of the two component theory, the four-fermion Hamiltonian still contains five arbitrary complex constants to be determined from experiments for each weak process, whereas if the weak interactions are truly « universal », we might expect only a single fundamental constant.

All these considerations indicate that a deeper insight is needed to understand the nature of parity non-conserving weak interactions. Although we do not believe that fundamental laws of physics can be derived from *a priori* considerations, invariance principles have guided us, in the past, to formulations of correct physical laws almost as often as they have misguided us. It is the purpose of this paper to make an attempt to understand from a single invariance principle the very processes in which the well-established invariance principles have been shown to fail.

We start with the following transformation property of the Dirac equation which was first noted by TIOMNO [9]. The Dirac equation [10]

$$[i\gamma_\mu(p_\mu - eA_\mu) + m]\psi = 0 \tag{1}$$

is invariant under the transformation

$$\psi \to \eta\gamma_5\psi\,, \qquad \overline{\psi} \to -\eta^*\overline{\psi}\gamma_5\,, \qquad m \to -m\,, \tag{2}$$

where

$$|\eta|^2 = 1\,. \tag{3}$$

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(4) S. B. TREIMAN and H. W. WYLD, Jr.: *Phys. Rev.*, **106**, 1320 (1957).

(5) F. EISLER, R. PLANO, N. SAMIOS, M. SCHWARTZ and J. STEINBERGER: *Phys. Rev.*, **107**, 324 (1957).

(6) J. A. MCLENNAN Jr.: *Phys, Rev.*, **106**, 821 (1957).

(7) K. M. CASE: *Phys. Rev.*, **107**, 307 (1957).

(8) S. BLUDMAN: *Proceedings of the Seventh Annual Rochester Conference* (to be published).

(9) J. TIOMNO: *Nuovo Cimento*, **1**, 226 (1955).

(10) We use Hermitian γ matrices with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. In particular our definition of γ_5 differs from that of Reference (11) by a facor of i.

This amounts to saying that the equation

$$[i\gamma_\mu(p_\mu - eA_\mu) - m]\psi' = 0 \tag{4}$$

is as good as the equation (1) and that Dirac might have discovered (4) instead of (1) in the very beginning. The two solutions ψ and ψ' in a given problem are related to each other by the simple interchange of « large » and « small » components.

So far there is no physics. We now ask: Why should the sign of m matter when the only relativistic requirement is of the form

$$m^2 = p_0^2 - |\boldsymbol{p}|^2 ,$$

where m occurs quadratically? Can the interaction Hamilton as well as the free field Hamiltonian be made invariant under mass reversal? These questions suggest a fundamental principle: *The four-fermion interaction responsible for a weak process must be invariant under the transformation* (2) *with the same* η *for every one of the four fields separately* (i.e. not simultaneously).

The consequences of this invariance principle are far-reaching. We now have a *unique* form of the four-fermion interaction

$$H_{\text{int.}} = G[(\overline{\psi}_2\gamma_\mu(1 \pm \gamma_5)\psi_1)(\overline{\psi}_4\gamma_\mu(1 \pm \gamma_5)\psi_3] + \text{h.c.} \tag{5}$$

provided that

$$\psi_1 \neq \psi_2 , \qquad \psi_3 \neq \psi_4 .$$

To see this consider, for instance, the bilinear product $\overline{\psi}_2 O_i(a + b\gamma_5)\psi_1$ where $\psi_1 \neq \psi_2$ and $O_i = 1$, γ_μ or $\gamma_\mu\gamma_\nu$. Appliyng the transformation (2) to ψ_1, we obtain from our invariance principle

$$\eta = \pm 1 , \qquad a = \pm b . \tag{6}$$

Similarly we obtain by applying (2) to ψ_2

$$O_i = \gamma_\mu \quad \text{only} . \tag{7}$$

Thus the unique form (5) results from our fundamental invariance requirement. Our interaction Hamiltonian (5) contains equal amounts of V and A in the β-decay language, it is necessarily invariant under time reversal but violates parity regardless of whether or not the neutrino field is involved in the interaction, and it reduces to the two-component theory of the neutrino when ψ_3, for instance, is ψ_ν.

The results we have obtained are identical to those obtained by Feynman and Gell-Mann who approached the same problem rather differently [11]. This is not too surprising. Their starting point is the second-order equation

$$\left[(p_\mu - eA_\mu)^2 - \frac{e}{2}\,\sigma_{\mu\nu}F_{\mu\nu}\right]\varphi = -\,m^2\varphi\,, \tag{8}$$

where the question of the sign of m simply does not enter. In (8) γ_5 commutes just as it is, whereas in the usual Dirac equation (1) we have to change the « sign » of the mass when we try to commute γ_5. Feynman and Gell-Mann proceed to seek a solution to (8) under the constraint

$$\chi = \gamma_5\chi$$

and they require that such a solution appear in their four-field Hamiltonian.

It ought to be emphasized that, if we had required invariance under the transformation (2) only for some of the fields, we would have obtained less stringent conditions [12]. Needless to say, Salam's neutrino-gauge transformation is a special case of ours, although in his formulation the role played by the vanishing mass of the neutrino is deliberately emphasized [1]. If we require invariance under the transformation (2) for the electron field only, we obtain Feynman's earlier theory [13]. Even in the days when no sophisticated physicists questioned parity non-conservation, Tiomno deduced that the β-decay Hamiltonian must consist of a linear combination of V and A *or* a linear combination of S, T and P by applying the transformation (2) *simultaneously* to both nucleon fields [9].

The sign of η (and hence the sign in (5) and (6)) cannot be determined on any *a priori* grounds. Rather we should *define* the « true » fermion field to be that field which makes the weak-interaction Hamiltonian invariant with $\eta = +1$, i.e. under

$$\psi \to \gamma_5\psi\,, \qquad \overline{\psi} \to -\,\overline{\psi}\gamma_5\,, \qquad m \to -\,m \tag{9}$$

so that the four-field interaction reads

$$H_{\text{int.}} = G[(\overline{\psi}_2\gamma_\mu(1+\gamma_5)\psi_1)(\overline{\psi}_4\gamma_\mu(1+\gamma_5)\psi_3] + \text{h. c.} \tag{10}$$

where ψ_1, ψ_2, ψ_3 and ψ_4 are all « true » fermion fields. For instance, if the relativistic positron from an allowed pure Fermi transition $(0 \to 0$, no) is

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(11) R. P. Feynman and M. Gell-Mann: to be published.

(12) S. Hori and A. Wakasa: *Nuovo Cimento*, **6**, 304 (1957).

(13) R. P. Feynman: *Proceedings of the Seventh Annual Rochester Conference* (to be published).

experimentally found to be right-handed, we say that the associated neutrino (which must be *left-handed* from angular momentum considerations) is a « true » fermion and that the positron is an anti-fermion. It is evident that lepton conservation in the two-component theory of the neutrino follows as a special case of our fermion conservation.

That this kind of definition of the fermion field is needed is rather obvious. Given a world and its anti-world, it is somewhat arbitrary to decide which one is the « true » world. Once we make this choice, however, the sense of right and left can be uniquely defined from parity experiments without recourse to any convention. Conversely, if we fix the sense of right and left by convention, the sense of « true » and « anti » can be unambiguously defined from the asymmetrical behavior of weakly interacting particles.

Suppose we require in (10) that either ψ_2 or ψ_1 (but not both) and either ψ_4 or ψ_3 (but not both) be electrically charged. In addition we require that if ψ_1 (or ψ_3) is a baryon field, ψ_2 (or ψ_4) must also be a baryon field. Then we can forbid processes such as

$$
(11)\qquad \left\{\begin{array}{ll}
\mu^- \to e^- + e^+ + e^- , & \mu^- + p \to e^- + p \\
K^+ \to \pi^+ + \nu + \bar{\nu} , & K^+ \to \pi^+ + \mu^+ + e^- \\
K^0 \to \mu^+ + e^- , & \text{etc.}
\end{array}\right.
$$

which to date have not been observed [14]. However, one may object that our selection rule is rather arbitrary since from (10) we can readily obtain

$$
H_{\text{i.t.}} = G\left[(\overline{\psi}_4\gamma_\mu(1+\gamma_5)\psi_1)(\overline{\psi}_2\gamma_\mu(1+\gamma_5)\psi_3)\right] + \text{h. c.}
$$

just by the Fierz transformation [15]. To prevent this we introduce an intermediate *charged* boson field which is coupled to two fermions (only one of which is charged) with strength of the order $\sqrt{G}$, and is responsible for all weak processes. We require the conservation of baryon number and electric charge at every vertex. Then the processes (11) can be strictly forbidden. Speculations as to the properties of this boson are extremely interesting, but they are not the primary purpose of the present paper. Similar considerations have been made previously [11,16].

From Eqs. (1) and (3) it is evident that electrodynamics is invariant under our mass reversal transformation. Similarly in the Feynman-Gell-Mann ap-

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[14] This selection rule has no effect on the decay of hyperons with pion emission, since virtual strong interactions can alter the charge states.

[15] M. FIERZ: *Zeits. f. Phys.*, **104**, 553 (1937).

[16] S. OGAWA: *Prog. Theor. Phys*, **15**, 487 (1956). See also Reference [19] J. SCHWINGER: *Annals of Physics* (to be published).

proach, the use of χ in electrodynamics does not alter any physics, for the reason that no particles can be created or destroyed in the Feynman sense, i.e. if only photons can be allowed to be emitted or absorbed, the charged fermion always stays as it is in the space-time sense of Feynman even if it may go backward in time (13,17). A difficulty arises when we consider the $(\bar{p}\,n)\pi^-$ interaction. Here one fermion does change into another fermion, and if we require the mass reversal principle in our strongest sense used in deducing the structure of the four-fermion Hamiltonian, parity will be violated in strong interactions. In order that the $(\bar{p}\,n)\pi^-$ interaction be parity-conserving, we must apply the mass reversal transformation *simultaneously* to the proton and neutron fields, as done by TIOMNO (9).

The essential point is that *our* mass reversal principle cannot be satisfied for strong interactions (i.e. strangeness-conserving meson-baryon interactions) although those interactions do satisfy Tiomno's weaker mass reversal principle. In strong interactions the *relative* signs of baryon masses are important, so that whenever we apply $m_p \to -m_p$ we must also apply *simultaneously* $m_n \to -m_n$, whereas in the electromagnetic and weak interactions the relative signs of fermion masses are left completely arbitrary. Our stronger invariance principle applicable only to the electromagnetic and weak interactions arises precisely from this arbitrariness. The apparent symmetry imposed by our stronger mass reversal invariance is destroyed when we « switch on » strong meson-baryon interactions. In fact, this point has been already noted by SUDARSHAN and MARSHAK who arrived at the same conclusions as ours still from another point of view (18). Their « chirality » invariance which is fully equivalent to our mass reversal invariance holds in so far as we do not switch on strong interactions. Such a view seems to be at variance with the currently accepted idea that the weaker are the interactions, the more symmetries are violated (19). Strong meson-baryon interactions are invariant under parity and charge conjugation, but they violate our mass reversal invariance; weak interactions, while violating parity and charge conjugation invariances, do preserve our mass reversal invariance. The electromagnetic interaction is invariant under all three parity, charge conjugation and mass reversal.

So much for the theoretical aspects of the problem. Now we may ask: Does our theory correspond in any way to the state of affairs of our actual physical world? The evaluation of such a theory in the light of the current experimental results has been carried out in details for β-decay by SUDARSHAN

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(17) R. P. FEYNMAN: *Phys. Rev.*, 76, 749 (1949).

(18) E. C. G. SUDARSHAN and R. E. MARSHAK: *Padua-Venice International Conference on Mesons and Recently Discovered Particles* (1957).

(19) See e.g. T. D. LEE: *Lectures at Harvard University* (March, 1957); J. SCHWINGER: *Annals of Physics* (to be published).

and MARSHAK ([18]). Here we briefly sketch qualitatively an argument in favor of reducing the most general four-component β-decay Hamiltonian to the form (1) ([20]).

(I) From electron (positron) polarization experiments ([21]) we deduce

$$C_S = -C'_S\,, \quad C_V = C'_V\,, \quad C_A = C'_A \quad \text{and} \quad C_T = -C'_T\,.$$

(II) From the Wu-type asymmetry experiment ([22]) with pure Gamow-Teller transitions we also deduce $C_A = C'_A$, $C_T = -C'_T$.

(III) The absence of the Fierz interference is consistent with (I) and (II).

(IV) From the ^{46}Sc β-γ circular polarization experiment ([23]) and from (I) we deduce that $|\mathrm{Re}\,(C_S C_T^*)|$ or $|\mathrm{Re}\,(C_V C_A^*)|$ (or possibly both) is large. (However, this conclusion contradicts the neutron asymmetry experiment ([24])).

(V) The neutron and ^{19}Ne recoil experiments ([25,26]) are consistent with (IV).

(VI) The ^{35}A recoil experiment ([27]) shows $|C_V| \gg |C_S|$. Hence $|C_A| \gg |C_T|$ from (IV) and (V).

(VII) The ^{6}He recoil experiment ([28]) shows very definitely $|C_T| \gg |C_A|$.

Thus if (IV) and (V) are to be accepted, we may be tempted to conclude that either the ^{35}A or ^{6}He experiment is wrong. If we accept the ^{35}A experiment, we conclude that the β-decay Hamiltonian is predominantly V and A with

$$C_V = +C'_V\,, \qquad C_A = +C'_A$$

([20]) The coupling constants Ci and Ci' are defined as in T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956).

([21]) H. FRAUENFELDER, R. BOBONE, E. VON GOELER, N. LEVINE, H. R. LEWIS, R. N. PEACOCK, A. ROSSI and G. DE PASQUALI: *Phys. Rev.*, **106**, 386 (1957); M. DEUTSCH, B. GITTELMAN, R. W. BAUER, L. GRODZINS and A. W. SUNYAR: *Phys. Rev.*, **107**, 1733 (1957); N. B. KOLLER, A. SCHWARZSHILD, J. B. VISE and C. S. WU (to be published).

([22]) C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPES and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957).

([23]) F. BOEHM and A. H. WAPSTRA: *Phys. Rev.*, **107**, 1462 (1957).

([24]) M. T. BURGY, R. J. EPSTEIN, V. E. KROHN, T. B. NOVEY, S. RABOY, G. R. RINGO and V. L. TELEGDI: *Phys. Rev.*, **107**, 1731 (1957).

([25]) J. M. ROBSON: *Phys. Rev.*, **100**, 933 (1955).

([26]) D. R. MAXSON, J. S. ALLEN and W. K. JENTSCHKE: *Phys. Rev.*, **97**, 109 (1955); M. L. GOOD and E. J. LAUER: *Phys. Rev.*, **105**, 213 (1957).

([27]) W. B. HERRMANSFELDT, D. R. MAXSON, P. STAHELIN and J. S. ALLEN: *Phys. Rev.*, **107**, 641 (1957).

([28]) B. M. RUSTAD and S. L. RUBY: *Phys. Rev.*, **97**, 991 (1955).

whereas if we accept the ⁶He experiment we say the S and T are dominant with

$$C_S = -C'_S, \qquad C_T = -C'_T.$$

From the β-decay experiments *alone* we have little reason *at the present* to assert which of the two recoil experiments is wrong, and either choice is consistent with the two-component theory of the neutrino.

However, from the π-μ-e sequence we are inclined towards favoring the ^{35}A experiment. Experimentally it was observed that the *positron* from the μ^+-decay is right-handed, which is contrary to the assignment $C_i = -C'_i$ if the positron is emitted preferentially backward with respect to the muon *monentum* [29]. Moreover, if we choose the ^{35}A experiment, not only can we save the two-component theory with lepton conservation, but also we can make the weak interactions « universal ». Hence, there is some possibility that the Hamiltonian (10) *might* represent the actual state of affairs [30].

Both β-decay and muon decay involve neutrinos, and it would be nice if our theory could predict something definite for processes in which the neutrino plays no role. Consider the decay of Λ particles

$$\Lambda^0 \to p + \pi^- . \tag{12}$$

We can write down phenomenologically the angular distribution of pions from polarized spin $\frac{1}{2}$ Λ particles as

$$I \sim 1 + \alpha \langle \boldsymbol{\sigma}_\Lambda \rangle \cdot \hat{p}_\pi , \tag{13}$$

$\langle \boldsymbol{\sigma}_\Lambda \rangle$ in (13) is the polarization of the Λ particle at the time of its decay, and α is the asymmetry parameter inherent in the parity non-conserving Λ decay, which is given by

$$\alpha = \frac{2\,\mathrm{Re}\,(A_s A_p^*)}{|A_s|^2 + |A_p|^2},$$

where A_s and A_p are respectively the amplitudes for s and p wave emission of pions in the decay of Λ particles. Experimentally the results based on

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(29) G. Culligan, S. G. F. Frank, J. C. Kluyver and J. R. Hold: *Padua-Venice International Conference on Mesons and Recently Discovered Particles* (1957).

(30) It might be appropriate to remark here that the idea of fitting the β decay experiments and the π-μ-e sequence with V and A occurred to the present author several weeks before he was informed of the Feynman-Gell-Mann-Sudarshan-Marshak theory. However, this idea was rejected by Professor C. N. Yang on the ground that the ⁶He recoil experiment is one of the more reliable experiments. C. N. Yang (private communication).

353 decay events of Λ particles produced in π-p collisions at 990 MeV by the Berkeley bubble-chamber group [31] show

$$\alpha\langle\overline{\boldsymbol{\sigma}_\Lambda}\rangle = (0.44 \pm 0.11)\hat{n}\,,$$

where

$$\hat{n} = \frac{\boldsymbol{p}_{\text{in}} \times \boldsymbol{p}_\Lambda}{|\boldsymbol{p}_{\text{in}} \times \boldsymbol{p}_\Lambda|}\,.$$

Similar results are obtained by the Bologna, Columbia, Michigan and Pisa groups for Λ particles produced at Brookhaven [32]. Since it is not very likely that the *average* polarization of the Λ particles in the reaction

$$\pi^- + \mathrm{p} \to \Lambda^0 + \mathrm{K}^0$$

exceeds 60%, α must be rather close to unity.

Field-theoretically we may regard the decay process (12) as a two-step process as shown in Fig. 1. The circle represents the fundamental four-fermion interaction, and the rectangular box represents the annihilation of a baryon-anti-baryon pair. In this case we can calculate the asymmetry parameter α by treating the $(\bar{\Lambda}\,\mathrm{p})\pi$ interaction as though it were primary [33]. Such calculations have been done previously [34]. If the weak vertex involves equal amounts of V and A as our mass reversal invariance requires, we obtain $|\alpha| = 0.89$; for equal amounts of S and P we have $|\alpha| = 0.10$. Thus the observed large « up-down » asymmetry is consistent with our mass reversal invariance. One may say that virtual strong interactions other than the rectangular box in Fig. 1 will modify the value of the asymmetry parameter. This is a valid objection, but a similar situation arises in the β-decay case. Experimentally the comparative study of the life-time of light mirror nuclei

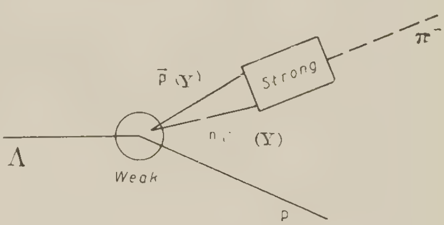

Fig. 1.

([31]) F. Crawford, M. Cresti, M. L. Good, K. Gottstein, E. M. Lyman, F. T Solmitz, M. L. Stevenson and H. K. Ticho: *UCRL*,-8008 (to be published).

([32]) The data compiled at the Venice-Padua Conference (1957) show $\alpha\langle\overline{\boldsymbol{\sigma}_\Lambda}\rangle\cdot\hat{n} = 0.46 \pm 0.14$ in excellent agreement with the Berkeley result.

([33]) S. Oneda, S. Hori and A. Wakasa: *Prog. Theor. Phys.*, **15**, 302 (1956).

([34]) J. J. Sakurai: *Phys. Rev.*, **108**, 491 (1957).

and neutron shows

$$\frac{C_{GT}^2}{C_F^2} = \frac{|C_A|^2 + |C'_A|^2 + |C_T|^2 + |C'_T|^2}{|C_S|^2 + |C'_S|^2 + |C_V|^2 + |C'_V|^2} = 1.3 \pm 0.1\,,$$

whereas our theory would predict precisely unity for this value if it were not for pions. As already remarked by FEYNMAN and GELL-MANN [11] this cannot be taken as evidence against our theory since virtual strong interactions can modify this ratio. For the same reason we do not take $|\alpha| = 0.89$ too seriously. Further, strong final-state interactions may change this value by an amount not more than 20% [35]. The essential point is that our theory predicts $|\alpha|$ of the order of unity and not of the order of 0.1 [36].

One of the important questions in our theory yet to be settled from experiments is whether the β-decay Hamiltonian is $V-A$ or $V+A$ in the conventional language. This is related to the question whether nucleons (or, for that matter, all baryons) are fermions or anti-fermions in the sense of Eq. (9). Suppose this is settled by looking into the sign of the VA interference in β-γ circular polarization experiments or in asymmetry (e.g. neutron) experiments. We may then ask whether the sign of α in the Λ decay is consistent with those β-decay experiments. If we study various polarization effects of the recoil proton in the Λ decay as discussed by GATTO [37] and by LEE and YANG [38], it may not be entirely impossible to determine the sign of α, which, in turn, serves as an independent assignment of fermion number to the Λ particle and nucleon.

Finally we discuss various decay modes of K particles. As far as the $K_{\pi 2}$ ($=\theta$) and $K_{\pi 3}$ ($=\tau$) modes are concerned, parity non-conservation here is no more mysterious than parity non-conservation in β-decay, which was already noted by CASE [7]. Our Hamiltonian (10) accounts for parity-non-conservation in both processes. As mentioned earlier some of the yet unobserved decay modes of K particles can be forbidden by introducing a charged intermediate boson field. The suppression of the K_{e2} ($\rightarrow e+\nu$) mode follows in the same manner as in the case of pion decay (which is not entirely satis-

(35) This can be estimated by a technique similar to that used by GATTO who showed that the observed large « up-down » asymmetry also violates invariance under charge conjugation. R. GATTO: *UCRL*-8009 (to be published).

(36) It is worth noting in this connection that speculations as to the asymmetry parameter α from the study of hypernuclei have led to values of $|\alpha|$ considerably smaller than 0.89. See, e.g. R. H. DALITZ: Appendix to « *K-Mesons and Hyperons* », *Reports on Progress in Physics* (to be published).

(37) R. GATTO: *UCRL*-3795.

(38) T. D. LEE and C. N. YANG: *Phys. Rev.* (to be published).

factory if the present experimental estimate is correct) ([39]). The branching ratio of the K_{e2} mode to $K_{\mu 2}$ mode turns out to be 1/40000. Note that although the K_{e3} mode is suppressed by postulating the dominance of V or A (depending on the relative parity of the Λ and the K), the decay mode K_{e3} ($\to \pi + e + \nu$) is fully allowed with A or V. The angular correlation experiment in the K_{e3} decay suggested by PAIS and TRIMAN ([40]) is extremely important, since a mixture of S (or P) will give a larger branching ratio for the K_{e2} to $K_{\mu 2}$: In particular, if our theory is correct, the decay configurations in which all three decay products are collinear are forbidden independently of the positron energy for the K_{e3} decay (but not for the $K_{\mu 3}$ decay). The energy spectrum of positrons from the K_{e3} discussed by FURUICHI *et al.* is not too indicative of the covariants involved, since there is no reason to believe that the « effective » coupling constant is independent of pion energy, although perturbation calculations suggest that it is ([41]). The longitudinal polarization of the μ^+ from the $K_{\mu 3}$ decay is *not* complete even for relativistic muons ([42]). The e^+ from the K_{e3} decay, on the other hand, is 100% right-handed for practically the whole range of the energy spectrum.

In this paper we have made an attempt to understand all weak processes from a single invariance principle. Some of the specific predictions of our theory are discussed with reference to our current empirical knowledge of weak interactions. Should future recoil experiments with pure Gamow-Teller transitions show that the tensor interaction is indeed absent, we might even be tempted to claim that our mass reversal invariance (or chirality invariance of SUDARSHAN and MARSHAK) is as fundamental as gauge invariance in electrodynamics. If otherwise, our approach is but an elegant symmetry argument devoid of any physical content, which, in this respect, is no better than many of the past fruitless attempts to deduce the form of the weak interaction Hamiltonian from *a priori* principles.

* * *

The present investigation is directly stimulated by conversations the author had with Professor R. E. MARSHAK, to whom he wishes to extend his sincere

([39]) S. LOKANATHAN and J. STEINBERGER: *Nuovo Cimento*, **2**, 151 (1955); H. L. ANDERSON and C. M. G. LATTES (to be published).

([40]) A. PAIS and S. B. TREIMAN: *Phys. Rev.*, **105**, 1616 (1957).

([41]) S. FURUICHI, T. KODAMA, S. OGAWA, Y. SUGAWARA, A. WAKASA and M. YONEZAWA: *Prog. Theor. Phys.*, **17**, 89 (1957).

([42]) J. WERLE: *Nuclear Physics*, **4**, 171 (1957); S. W. MACDOWELL (to be published).

thanks. The author also wishes to acknowledge his indebtedness to Dr. T. KINOSHITA for helpful discussions and to Professor F. J. DYSON for interesting comments.

RIASSUNTO (*)

Si cerca di costruire una teoria di tutti i processi deboli partendo da un unico principio d'invarianza. L'esigenza che l'hamiltoniana d'interazione debole sia invariante rispetto all'inversione del segno della massa nell'equazione di Dirac separatamente per *ogni* campo di fermioni conduce ad un'unica forma dell'interazione quadri-fermionica producente processi deboli. L'hamiltoniana contiene eguali quantità di V e A ed è necessariamente invariante rispetto all'inversione del tempo ma non invariante rispetto alla parità sia che il campo neutrinico intervenga o non nel fenomeno. (I nostri risultati sono identici a quelli ottenuti da FEYNMAN e GELL-MANN e da SUDARSHAN e MARSHAK seguendo metodi un po' differenti). La possibile esistenza di un campo bosonico intermedio *carico*, causa di tutti i processi deboli si discute con riferimento ad alcuni processi deboli finora non osservati. La maggior parte degli esperimenti sul decadimento β e sulla sequenza π-μ-e sono compatibili con le previsioni della nostra teoria, con l'importante eccezione dell'esperimento sul rinculo del ^{6}He. Si esaminano varie conseguenze della nostra teoria suscettibili di verifica sperimentale con speciale riguardo alle interazioni di decadimento degli iperoni e dei mesoni K.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Angular Correlation for Production and Decay of Λ^0.

D. BARONCINI

Istituto di Fisica dell'Università - Bologna

V. DE SABBATA

Istituto di Fisica dell'Università - Parma
Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

(ricevuto il 5 Novembre 1957)

Summary. — The angular correlation function for production and decay of the Λ^0 particles is obtained assuming spin $S=\frac{1}{2}$ for the Λ^0 and zero spin for the θ^0. Parity conservation and parity non-conservation in decay are discussed under the alternative hypothesis of either a definite parity for the Λ^0-θ^0 system or of parity doublets. Parity non-conservation in decay leads to anisotropy with respect to the direction perpendicular to the plane of production. This anisotropy does not appear if parity is conserved. In either case the existence of parity doublets shows an anisotropic distribution in the plane of production, which involves odd powers of $\cos\vartheta$ (ϑ being the angle between the outgoing π^- and the Λ^0). If no parity doublets are assumed in production the angular distribution for the decay angle ϑ is in even powers of $\cos\vartheta$. Since the highest powers of $\cos\vartheta$ are determined by the value of the spin S of the Λ^0 particle, the angular distribution for decay can be indicative for the value S. Comparison is made with the experimental curves obtained in Bologna.

1. – Introduction.

As is well-known, the process

$$\pi^- + \mathrm{p} \to \Lambda^0 + \theta^0 \tag{a}$$

gives rise to unstable particles, Λ^0 and θ^0, which decay most frequently as

$$\begin{cases} \Lambda^0 \to \mathrm{p} + \pi^- \\ \theta^0 \to \pi^+ + \pi^- . \end{cases} \tag{b}$$

We aim to investigate whether from angular correlation of production (a) and decay (b) it is possible to obtain information about spin and parity of the Λ^0 particle.

For this purpose we calculate the angular correlation function under the hypothesis that the production takes place with conservation of parity, and we discuss the possible cases of production of the Λ^0-θ^0 system in states of definite parity and the case of production in states of parity doublets [1]. We calculate the angular correlation function for decay with parity conservation and for decay with no conservation of parity.

The technique of the density matrix [2,3] has been used without any assumption on the properties of the interactions among the particles in either process (a) and (b).

As it is well-known the angular correlation function is defined in terms of the density matrices for each of the processes involved. One has:

$$W = \sum E_{\text{prod}} E_{\text{dec}\ \Lambda^0} E_{\text{dec}\ \theta^0}\,,$$

where W is a function of the angles of production and decays and the sum is performed over all the variables of the intermediate states.

In Sect. **2** we give an outline of the calculation of the density matrices and of the angular correlation function under the assumption of spin zero for the θ^0 particle and of spin $S=\frac{1}{2}$ for the Λ^0 particle.

In Sect. **3** we discuss the general features of the angular correlation function in connection with the conservation of parity and with no conservation of parity in the decay, related in each case to the assumption of either a definite parity or of parity doublets in the production.

In Sect. **4** we give detailed expressions of the angular distribution for the production. Here we assume the Λ^0-θ^0 system to be produced in s- and p-waves, as it seems to be in agreement with experimental results.

In Sect. **5** the angular correlation function is obtained explicitly for each of the cases discussed in Sect. **3**.

Finally we discuss our results with the experimental curves which have been obtained in Bologna and Pisa.

2. – The density matrix.

We calculate the angular correlation function for production and decay of Λ^0-θ^0 particles, making use of the method of the density matrix [2,3].

No assumption is made on the type of interaction. The calculation is per-

(1) T. D. Lee and C. N. Yang: *Phys. Rev.*, **102**, 290 (1956).

(2) L. C. Biedenham and M. E. Rose: *Rev. Mod. Phys.*, **25**, 729 (1953).

(3) S. Hori: *Prog. Theor. Phys.*, **16**, 189 (1956).

formed under the general assumptions of conservation of the total angular momentum. Conservation of the parity is assumed for the production process. Conservation and no-conservation of parity for the decay are investigated separately.

We analyze the production in the center of mass system of the Λ^0-θ^0, p-π^- and indicate with $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ the directions of the relative momenta of the Λ^0-θ^0 and of the p-π^- respectively.

The transition from the initial system P-π^- in the spin state μ, to the produced system Λ^0-θ^0 in a spin state m_s in the direction $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ respectively can be written as

$$(1) \qquad \langle m_s, \boldsymbol{p}_1 | \mu, \boldsymbol{p}_2 \rangle = \sum_{\substack{l_1 l_2 m_1 m_2 \\ J M_J}} a(J l_1 l_2)\, Y_{l_1}^{m_1 *}(\boldsymbol{p}_1) \langle S l_1 m_s m_1 | J M_J \rangle \langle \tfrac{1}{2} l_2 \mu m_2 | J M_J \rangle\, Y_{l_2}^{m_2}(\boldsymbol{p}_2)\,,$$

where J = is the total angular momentum,
M_J = its z-component,
l_1 = is the relative orbital angular momentum of the Λ^0-θ^0,
m_1 = the z-component of l_1,
l_2 = is the relative orbital angular momentum of p-π^-,
m_2 = the z-component of l_2,
S = the spin of the Λ^0 particle,
m_s = its z-component,
μ = the z-component of the spin of the bombarded proton.

The coefficient $a\ (J, l_1, l_2)$ involves the special properties of the interaction in production. In writing (1) we have assumed spin zero for the θ^0 particle.

It is convenient to refer to the z-axis as to the direction $\boldsymbol{p}_1$ and to work out the density matrix for production in terms of the angle ω between $\boldsymbol{p}_2$ and $\boldsymbol{p}_1$.

The density matrix for production is then derived after averaging over the spin of the initial proton as:

$$(2) \qquad E_{\text{prod}}(m_s, m_s' \boldsymbol{p}_2) = \tfrac{1}{2} \sum_{\mu} \langle m_s \boldsymbol{p}_1 | \mu \boldsymbol{p}_2 \rangle \langle \mu \boldsymbol{p}_2 | m_s' \boldsymbol{p}_1 \rangle =$$

$$= \tfrac{1}{2} \sum_{\substack{l_1 l_1' l_2 l_2' \\ J J' \\ M_J M_J' L_2}} (-)^{J'-\frac{1}{2}} [(2l_1+1)(2l_1'+1)(2l_2+1)(2l_2'+1)(2J'+1)]^{\frac{1}{2}} a(J l_1 l_2)\, a^*(J' l_1' l_2') \cdot$$

$$\cdot \langle S l_1 m_s 0 | J M_J \rangle \langle s l_1' m_s' 0 | J' M_J' \rangle \langle J' L_2 M_J' M_2 | J M_J \rangle \langle l_2' l_2 00 | L_2 0 \rangle \cdot$$

$$\cdot W(J' l_2' J l_2;\ \tfrac{1}{2} L_2)\, Y_{L_2}^{M_2}(\boldsymbol{p}_2)\,,$$

where W denotes the well-known Racah coefficient, $\boldsymbol{L}_2 = \boldsymbol{l}_2 + \boldsymbol{l}'_2$ and M_2 is the z-component of L_2. In obtaining (2) one has to keep in mind conservation of the total angular momentum.

In a similar manner one can obtain the density matrix for the decay of the Λ^0 particle. We indicate with $\boldsymbol{p}_3$ the direction of the relative momentum of the decay products p-π^- in the rest system of the Λ^0 particle, and with ϑ the angle between $\boldsymbol{p}_3$ and the z-axis. We denote with φ the angle between the production plane and the decay plane. In this frame of reference (Fig. 1) the density matrix for decay is given by:

$$(3)\qquad E_{\Lambda^0}(m_s, m'_s\boldsymbol{p}_3) = (-)^{S-\frac{1}{2}}(2S+1)^{\frac{1}{2}} \sum_{ll'LM_L} [(2l+1)(2l'+1)]^{\frac{1}{2}} \lambda_{ll'} \cdot \\ \cdot \langle SLm_sM_L | Sm'_s\rangle \langle ll'00 | L0\rangle W(SlSl'; \tfrac{1}{2}L)\, Y_L^{M_L}(\boldsymbol{p}_3)\,,$$

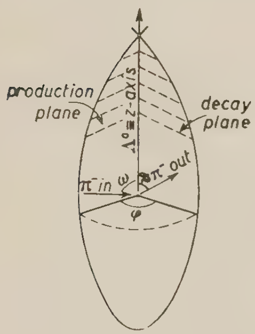

Fig. 1. – Frame of reference.

where $\boldsymbol{p}_3$ is described by the variables ϑ, φ, $\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{l}'$ and M_L is the z-component of L.

Here λ is a coefficient which depends on the interaction for the decay and has obviously the meaning of the reciprocal of a half-life: λ is unique if we confine ourselves to a decay mode in which parity is conserved and if the Λ^0 particle has a well defined parity.

A similar expression for the decay of the θ^0 particle reduces to a constant factor on account of the zero spin of the θ^0.

The angular correlation function is therefore given by:

$$(4)\qquad W(\boldsymbol{p}_2, \boldsymbol{p}_3) = \sum_{m_sm'_s} E_{\text{prod}}(m_sm'_s\boldsymbol{p}_2)\, E_{\Lambda^0}(m_sm'_s\boldsymbol{p}_3)\,.$$

Inserting (2) and (3) into (4) one has the explicit expression for $W(\boldsymbol{p}_2\boldsymbol{p}_3)$ in terms of the variables $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, i.e. of the angles ω, ϑ, φ

$$(5)\qquad W(\boldsymbol{p}_2, \boldsymbol{p}_3) = \sum_{l_2l'_2L_2} [(2l_2+1)(2l'_2+1)]^{\frac{1}{2}} \langle l_2l'_200 | L_20\rangle \cdot \\ \cdot \sum_{l_1l'_1JJ'} [(2J'+1)(2l_1+1)(2l'_1+1)]^{\frac{1}{2}}\, a(Jl_1l_2)a^*(J'l'_1l'_2)(-)^{J-\frac{1}{2}} \cdot \\ \cdot W(J'l'_2Jl_2; \tfrac{1}{2}L_2) \sum_{ll'L} [(2l+1)(2l'+1)(2S+1)]^{\frac{1}{2}} \lambda_{ll'} (-)^{S-\frac{1}{2}} \langle ll'00 | L0\rangle W(SlSl'; \tfrac{1}{2}L) \cdot \\ \cdot \sum_{m_sm'_s} \langle SLm_sM_L | Sm'_s\rangle \langle J'L_2m'_sM_2 | Jm_s\rangle \langle Sl_1m_s0 | Jm_s\rangle \cdot \\ \cdot \langle Sl'_1m'_s0 | J'm'_s\rangle\, Y_{L_2}^{M_2}(\boldsymbol{p}_2)\, Y_L^{M_L}(\boldsymbol{p}_3)\,.$$

We remark that in (5) the first two sums

$$(6)\qquad \sum_{l_2 l_2' L_2} \langle l_2 l_2' 00 | L_2 0\rangle [(2l_2+1)(2l_2'+1)]^{\frac{1}{2}} \sum_{l_1 l_1' J J'} [(2l_1+1)(2l_1'+1)(2J'+1)]^{\frac{1}{2}} \cdot$$
$$\cdot\, a(Jl_1l_2)\, a^*(J'l_1'l_2')(-)^{J-\frac{1}{2}}\, W(J'l_2'Jl_2;\tfrac{1}{2}L_2)$$

contain purely the kinematics in the production.

In a similar way the sum:

$$(7)\qquad \sum_{ll'L} \langle ll'00 | L0\rangle [(2l+1)(2l'+1)(2S+1)]^{\frac{1}{2}} \lambda_{ll'}\, W(SlSl';\tfrac{1}{2}L)$$

involves the kinematics for the decay only.

The last part

$$(8)\qquad \sum_{m_s m_s'} \langle SLm_sM_L | Sm_s'\rangle \langle J'L_2m_s'M_2 | Jm_s\rangle \langle Sl_1m_s0 | Jm_s\rangle \cdot$$
$$\cdot \langle Sl_1'm_s'0 | J'm_s'\rangle\, Y_{L_2}^{M_{L_2}}(\boldsymbol{p}_2)\, Y_L^{M_L}(\boldsymbol{p}_3)$$

is the actual angular correlation factor. Because of conservation of the z-component of the total angular momenta,

$$M_2 = -M_L\,,$$

so that for the case in which $S=\frac{1}{2}$ (and therefore $m_s=\pm\frac{1}{2}$) one can write (8) as:

$$(9)\qquad \sum_{m_s=m_s'} \langle SLm_s0 | Sm_s\rangle \langle J'L_2m_s'0 | Jm_s\rangle \langle Sl_1m_s0 | Jm_s\rangle \langle Sl'm_s'0 | J'm_s'\rangle\, Y_{L_2}^0(p_2)\, Y_L^0(p_3) +$$
$$+ \sum_{m_s'=-m_s} \langle SLm_s - 2m_s | S - m_s\rangle \langle J'L_2 - m_s 2m_s | Jm_s\rangle \cdot$$
$$\cdot \langle Sl_1m_s0 | Jm_s\rangle \langle Sl_1' - m_s0 | J' - m_s\rangle\, Y_{L_2}^{2m_s}(\boldsymbol{p}_2)\, Y_L^{-2m_s}(\boldsymbol{p}_3)\,,$$

where the two contributions arising from $M_L=0$ and $M_L\neq 0$ are now separated. One can write shortly for (9) the expression:

$$(9')\qquad \langle Sl_1\tfrac{1}{2}0 | J\tfrac{1}{2}\rangle \langle Sl_1'\tfrac{1}{2}0 | J'\tfrac{1}{2}\rangle\, F_L(L_2 | JJ' | l_1l_1')\,,$$

where, on account of a well-known property of the C.G. coefficients $F_L(L_2 | JJ' | l_1l_1')$ is defined as:

$$(10)\qquad F_L(L_2 | JJ' | l_1l_1') = \langle J'L_2\tfrac{1}{2}0 | J\tfrac{1}{2}\rangle \langle SL\tfrac{1}{2}0 | S\tfrac{1}{2}\rangle\, Y_{L_2}^0(\boldsymbol{p}_2)\, Y_L^0(\boldsymbol{p}_3)\,[1+(-)^{\alpha}] +$$
$$+ (-)^{S-J+L_2+l_1'} \langle J'L_2\tfrac{1}{2} - 1 | J - \tfrac{1}{2}\rangle \langle SL\tfrac{1}{2} - 1 | S - \tfrac{1}{2}\rangle \cdot$$
$$\cdot [Y_{L_2}^1(\boldsymbol{p}_2)\, Y_{L_2}^{-1}(\boldsymbol{p}_3) + (-)^{\alpha}\, Y_{L_2}^{-1}(\boldsymbol{p}_2)\, Y_L^1(\boldsymbol{p}_3)]$$

and where $\alpha = L_2 + l_1 + l_1' + L$.

The angular correlation function can be written now in a more compact form as:

$$(5') \qquad W(\boldsymbol{p}_2, \boldsymbol{p}_3) = \sum_{l_2 l'_2 L_2} [(2l_2+1)(2l'_2+1)]^{\frac{1}{2}} \langle l_2 l'_2 00 | L_2 0\rangle \cdot$$

$$\cdot \sum_{l_1 l'_1 J J'} [(2J'+1)(2l_1+1)(2l'_1+1)]^{\frac{1}{2}}\, a(J l_1 l_2)\, a^*(J' l'_1 l'_2)(-)^{J'-\frac{1}{2}} W(J' l'_2 J l_2; \tfrac{1}{2} L_2)\cdot$$

$$\cdot \langle S l_1 \tfrac{1}{2} 0 | J \tfrac{1}{2}\rangle \langle S l'_1 \tfrac{1}{2} 0 | J' \tfrac{1}{2}\rangle \sum_{l l' L} [(2l+1)(2l'+1)(2S+1)]^{\frac{1}{2}} \langle l l' 00 | L0\rangle \cdot$$

$$\cdot \lambda_{ll'} W(SlSl'; \tfrac{1}{2} L)\, F_L(L_2 \; JJ' \; l_1 l'_1)\,.$$

3. – Generalities of the angular correlation.

The angular dependance of $W(\boldsymbol{p}_2, \boldsymbol{p}_3)$ is strictly related to the value of

$$\alpha = L_2 + l_1 + l'_1 + L$$

namely to the addition modes of the angular momenta in the whole process of production and decay. Clearly, the value of α is determined by the parity of the Λ^0-θ^0 system which is assumed to be produced with conservation of parity; α is also determined by the alternative assumptions of parity conservation or parity non-conservation in the decay.

We remark that if the decay takes place with conservation of parity and if the Λ^0 particle is assumed to have a well definite intrinsic parity (*) there is only one possible value for the relative angular momentum of the p-π^- system, namely

$$\text{if the } \Lambda^0 \text{ has} + \text{parity}, \qquad l = l' = 1\,,$$

$$\text{if the } \Lambda^0 \text{ has} - \text{parity}, \qquad l = l' = 0\,.$$

In either cases $L = 0$ and therefore $\alpha =$ even.

In this case one must expect an isotropic distribution of the outgoing p-π^- pair with respect to the z-axis and an isotropic distribution of the decay plane with respect to the production plane.

The function F as given in (10) reduces here to:

$$(10') \qquad F_0(L_2 | JJ' | l_1 l'_1) = 2\langle J' L_2 \tfrac{1}{2} 0 | J \tfrac{1}{2}\rangle\, Y^0_{L_2}(\boldsymbol{p}_2)\,.$$

(*) We remark that this is a more restrictive condition than to assume a definite parity for the Λ^0-θ^0 system in production.

The angular correlation function as given in (5′) becomes:

$$W(\boldsymbol{p}_2\boldsymbol{p}_3) = \sum_{l_2 l_2' L_2} [(2l_2+1)(2l_2'+1)]^{\frac{1}{2}} \langle l_2 l_2' 00 \,|\, L_2 0\rangle \cdot \tag{11}$$
$$\cdot \sum_{l_1 l_1' J J'} [(2J'+1)(2l_1+1)(2l_1'+1)]^{\frac{1}{2}} a(Jl_1l_2)\, a^*(J'l_1'l_2')(-)^{J'-\frac{1}{2}}\, W(J'l_2'Jl_2;\tfrac{1}{2}L_2)\cdot$$
$$\cdot 2\sum_{ll'} [(2l+1)(2l'+1)(2S+1)]^{\frac{1}{2}} \lambda_{ll'} \langle ll'00\,|\,L0\rangle\, W(SlSl';\tfrac{1}{2}0)\cdot$$
$$\cdot \langle Sl_1 \tfrac{1}{2} 0\,|\,J\tfrac{1}{2}\rangle \langle Sl_1' \tfrac{1}{2} 0\,|\,J'\tfrac{1}{2}\rangle \langle J'L_2 \tfrac{1}{2} 0\,|\,J\tfrac{1}{2}\rangle\, Y^0_{L_2}(\boldsymbol{p}_2)\,.$$

If there is no conservation of parity in the decay and still Λ^0 has a definite parity, one can expect either cases $l=l'=1$, $l=l'=0$ to occur ($L=0$) and at the same time the cases $l=0$, $l'=1$ and $l=1$, $l'=0$ ($L=1$).

The last two cases have obviously the meaning of interference of angular momenta as an effect of interference of parity.

Therefore both combinations occur:

$$\alpha = \text{even}\,, \qquad L=0\,; \qquad\qquad \alpha = \text{odd}, \quad L=1.$$

The parity interference terms ($L=1$) give rise to a non-isotropic term ($M_L \neq 0$) in the decay angles ϑ, φ for which one has:

$$F_1^-(L_2\,|\,JJ'\,|\,l_1 l_1') = (-)^{\frac{1}{2}-J+L_2+l_1'} \langle J'L_2 \tfrac{1}{2} -1\,|\,J-\tfrac{1}{2}\rangle \langle S1\tfrac{1}{2}-1\,|\,S-\tfrac{1}{2}\rangle\,\cdot \tag{12}$$
$$\cdot [Y^1_{L_2}(\boldsymbol{p}_2)\, Y^{-1}_L(\boldsymbol{p}_3) - Y^{-1}_{L_2}(\boldsymbol{p}_2)\, Y^1_L(\boldsymbol{p}_3)]\,.$$

At the same time the $L=0$ terms give an isotropic contribution of the form (10′).

The angular correlation function will contain therefore both isotropic and non-isotropic contributions.

If we now assume the Λ^0-θ^0 to be produced in states of opposite parity, for the production we expect both combinations.

$$l_2 + l_1 + l_2' + l_1' = \text{even}$$
$$l_2 + l_1 + l_2' + l_1' = \text{odd}$$

dependently wether the parity doublets interfere (odd) or do not interfere (even). In other words parity doublets in production lead to either.

$$L_2 + l_1 + l_1' = \text{even} \qquad \text{or to} \qquad L_2 + l_1 + l_1' = \text{odd}\,.$$

Conservation of parity for the decay leads to:

$$l + l' = \text{even in the first case} \qquad (L = 0)\,,$$

$$l + l' = \text{odd in the second case} \qquad (L = 1)\,.$$

We conclude that in this case $\alpha = L_2 + L + l_1 + l_1' = \text{even}$. When $L = 0$ one has again the contribution as given by (11). When $L = 0$ one has the additional contribution arising from:

$$(13) \qquad F_1^+(L_2\,|JJ'\,|l_1 l_1') = 2\langle J' L_2 \tfrac{1}{2} 0\,|J\tfrac{1}{2}\rangle\langle S 1 \tfrac{1}{2} 0\,|S\tfrac{1}{2}\rangle\, Y^0_{L_2}(\boldsymbol{p}_2) +$$
$$+ (-)^{\frac{1}{2}-J+L_2+l_1'}\langle J' L_2 \tfrac{1}{2} -1\,|J-\tfrac{1}{2}\rangle\langle S 1\tfrac{1}{2} -1\,|S-\tfrac{1}{2}\rangle\,[Y^1_{L_2}(\boldsymbol{p}_2)\,Y^{-1}_L(\boldsymbol{p}_3) + Y^{-1}_{L_2}(\boldsymbol{p}_2)\,Y^1_L(\boldsymbol{p}_3)].$$

This shows for the decay an anisotropic distribution in the angles ϑ, φ on the production plane.

Assume now that Λ^0-θ^0 are produced in parity doublets and moreover that the decay takes place without conservation of parity. This case associates both $L = 0, 1$ to each of the choices $L_2 + l_1 + l_1' = \text{even, odd}$. One has therefore $\alpha = \text{even}$, $\alpha = \text{odd}$ at the same time with $L = 0$, $L = 1$.

(One can see however from (10) that the case with $\alpha = \text{odd}$ gives contribution different from zero only for $M_L = 0$ and that $\alpha = \text{odd}$ and $L \neq 0$ does not give any contribution).

The angular correlation function will show both types of anysotropies for parity doublets ($\cos\vartheta$ and $\sin\vartheta\cos\varphi$) and for parity non-conservation ($\sin\vartheta\sin\varphi$).

In the following table we give a resumé of the results for the decay angular distribution.

TABLE I.

	Parity conservation				Parity non-conservation			
	α	L_2+ $+l_1+l_1'$	L	Angular dependence	α	L_2+ $+l_1+l_1'$	L	Angular dependence
No doublets	even	even	0	isotropic	even	even	0	isotropic
					even	even	1	$\lvert M_L\rvert = 1$ $\sin\vartheta\sin\varphi$
Parity doublets	even	even	0	isotropic	even	even	0	isotropic
	even	odd	1	$M_L = 0$ $\cos\vartheta$	even	odd	1	$M_L = 0$ $\cos\vartheta$ $\lvert M_L\rvert = 1$ $\sin\vartheta\cos\varphi$
				$\lvert M_L\rvert = 1$ $\sin\vartheta\cos\varphi$	odd	even	1	$\lvert M_L\rvert = 1$ $\sin\vartheta\sin\varphi$

4. – Angular distribution for production.

In view of possible comparison with experimental results for the production of Λ^0-θ^0 we shall give a detailed expression for the production cross-section:

$$(14)\qquad \sigma(\omega) \sim \sum_{m_s} E_{\text{prod}}(m_s m_s \boldsymbol{p}_2) = \sum_{m_s} \sum_{l_2 l'_2 L_2} [(2l_2+1)(2l'_2+1)]^{\frac{1}{2}} \langle l_2 l'_2 00 | L_2 0 \rangle \cdot$$
$$\cdot \sum_{l_1 l'_1 J J'} [(2J'+1)(2l_1+1)(2l'_1+1)]^{\frac{1}{2}} (-)^{J'-\frac{1}{2}} a(J l_1 l_2) a^*(J' l'_1 l'_2) W(J' l'_2 J l_2; \tfrac{1}{2} L_2) \cdot$$
$$\cdot \langle \tfrac{1}{2} l_1 m_s 0 | J m_s \rangle \langle \tfrac{1}{2} l'_1 m_s 0 | J' m_s \rangle Y^0_{L_2}(\boldsymbol{p}_2)\,.$$

As pointed out previously, we shall deal with $S=\frac{1}{2}$ and zero spin for the θ^0. We calculate (14) assuming s- and p-waves in the Λ^0-θ^0 system: the values for L_2 are 0, 1, 2. We obtain the angular distribution of (14) of the form:

$$(14')\qquad a + b\cos\omega + c\cos^2\omega\,.$$

This angular distribution is qualitatively in agreement with experimental curves which we show in Fig. 2, taking care of the fact that obviously is $b<0$ while $a, c>0$.

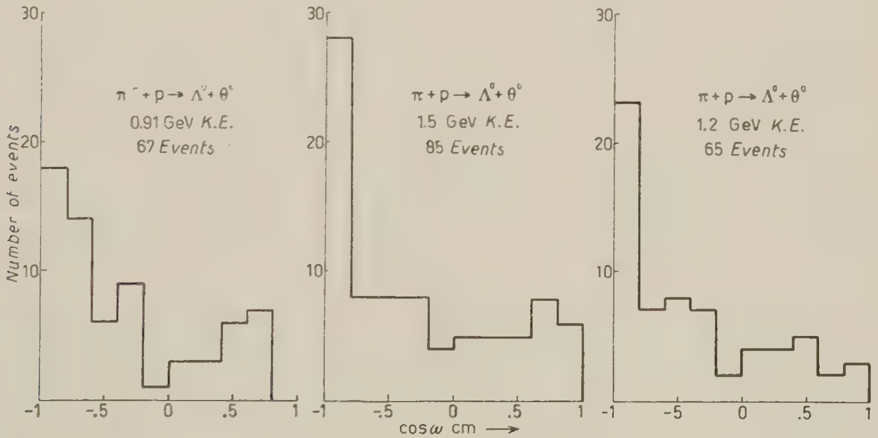

Fig. 2. – Experimental curves for production: a) at .91 GeV; b) at 1.2 GeV; c) at 1.5 GeV.

The coefficients a, b, c depend on the properties of the interaction, which we are not going to investigate in detail, and on the partial waves of the ingoing system p-π^-: these are related to the parity of the Λ^0-θ^0 system.

We can in fact distinguish two possibilities:

a) The Λ^0-θ^0 is emitted with a positive intrinsic parity. In this case the ingoing system shows, *S*-, *P*-, *D*-waves and $l_2+l_1=l_2'+l_1'=\text{odd}$, so that

$$L_2+l_1+l_1'=\text{even}\,.$$

The explicit expression for (14) is

$$(14'')\qquad I_+(\omega)=2\{\tfrac{1}{2}[101,101]+\tfrac{1}{2}[110,110]+\tfrac{1}{2}[312,312]-[110,312]\}+\\ +2\{[101,110]+2[312,101]\}\cos\omega+\\ +6\{[110,312]+\tfrac{1}{2}[312,312]\}\cos^2\omega\,,$$

where we indicate for brevity the real quantity:

$$a(Jl_1l_2)a^*(J'l_1'l_2')+a^*(Jl_1l_2)a(J'l_1'l_2')=[2Jl_1l_2,\,2J'l_1'l_2']\,.$$

b) The Λ^0-θ^0 system is emitted with a negative intrinsic parity. The ingoing system shows now only *S*- and *P*-waves and

$$l_1+l_2=l_1'+l_2'=\text{even}\,,$$

with

$$L_2+l_1+l_1'=\text{even}\,.$$

The expression for (14) is here:

$$(14''')\qquad I_-(\omega)=2\{\tfrac{1}{2}[100,100]+\tfrac{1}{2}[111,111]+\tfrac{1}{2}[311,311]-[111,311]\}+\\ +2\{[100,111]+2[100,311]\}\cos\omega+\\ +6\{[111,311]+\tfrac{1}{2}[311,311]\}\cos^2\omega\,.$$

As expectable, the angular dependence on ω is in either case the same. In both cases there appear six different coefficients (one of which can be determined by suitable normalization). Since the experimental curve shows a $\cos^2$-term, both expansions (14″) and (14‴) are qualitatively satisfactory. Any quantitative comparison of (14″) or (14‴) with the experimental curves is impossible, because so far we know very little about the six quantities involved in their expansions.

One might remark that if there were reasons for excluding from the production *D*-waves for the ingoing system p-π^-, one could decide about the relative parity of the system. This is not however the case, and we shall only analyze $I_\pm(\omega)$ under a few simple assumptions, and precisely as follows:

S- and P-waves with equal amplitudes (Fig. 3a)

S- and $P_{\frac{1}{2}}$ waves with amplitudes twice as large as $P_{\frac{3}{2}}$ (Fig. 3b)

S-wave with amplitude twice as large as P-waves (Fig. 4a)

P-waves with amplitude twice as large as S-wave (Fig. 4b)

$P_{\frac{1}{2}}$ wave with amplitude twice as large as S and $P_{\frac{3}{2}}$ (Fig. 5a)

$P_{\frac{3}{2}}$ wave with amplitude twice as large as S and $P_{\frac{1}{2}}$ (Fig. 5b)

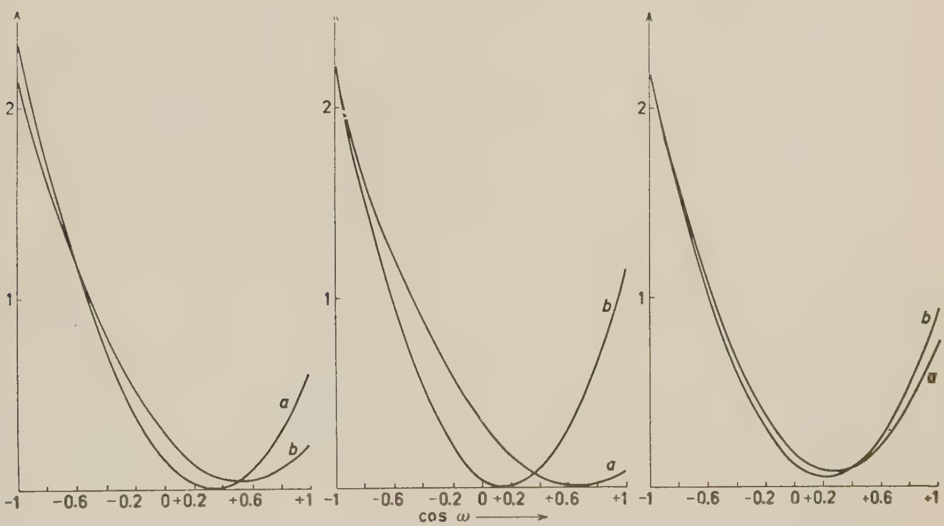

Fig. 3, 4 and 5. – Calculated angular distribution for production.

Fig. 3. a) S- and P-waves with equal amplitudes; b) S- and $P_{\frac{1}{2}}$-waves with amplitudes twice as large as $P_{\frac{3}{2}}$.

Fig. 4. a) S-wave with amplitude twice as large as P-waves; b) P-waves with amplitudes twice as large as S-waves.

Fig. 5. a) $P_{\frac{1}{2}}$-wave with amplitude twice as large as S and $P_{\frac{3}{2}}$; b) $P_{\frac{3}{2}}$-wave with amplitude twice as large as S and $P_{\frac{1}{2}}$.

All these curves have been normalized to 0.6 mb.

Before we compare these results with the experimental curves (Fig. 2), let us discuss the case in which the Λ^0-θ^0 system is produced in states of opposite parity. In this case, (14) will show cross terms of even-odd parity:

$$l_1 + l_2 = \text{odd} \qquad l_1' + l_2' = \text{even}$$

$$l_1 + l_2 = \text{even} \qquad l_1' + l_2' = \text{odd}$$

corresponding to

$$L_2 + l_1 + l_1' = \text{odd}$$

and at the same time all the combinations leading to $L_2+l_1+l_1'=$ even as given by the two choices a) and b). The selection for L_2 is however still the same, so that, again, the angular dependence on ω is still of the form

$$A + B\cos\omega + C\cos^2\omega\,.$$

The cross terms of even-odd parity give however, non-zero contribution only if the production cross-section is evaluated for Λ^0 particle polarized along the z-axis: one has

$$\begin{aligned}(15)\qquad E_{\text{prod}}(m_s, m_s) &= I_{ms}(\omega) = \tfrac{1}{2}[I_+(\omega) + I_-(\omega)] + \\ &+ (-)^{m_s-\frac{1}{2}}\{[101, 111] + [100, 110] - [100, 312] - [101, 311] + \\ &+ ([100, 101] + [110, 111] + 2[110, 311] + 2[111, 312] + \tfrac{2}{5}[311, 312])\cos\omega + \\ &+ (3[100, 312] + 3[101, 311])\cos^2\omega\,.\end{aligned}$$

Summation over the spin-components as in (14) gives:

$$(15')\qquad I(\omega) = I_+(\omega) + I_-(\omega)\,,$$

where the cross-terms of even-odd parity cancel.

From (15′) one sees easily that the parity doublets do affect the angular dependence of the production cross-section. We therefore compare any-one of our angular distributions (for no polarization of the Λ^0 along the z-axis) with the curves in Fig. 2.

In each of the six curves calculated one has to assume that the $\cos\omega$ coefficient has negative sign, in order to take into account the fact that the Λ^0 is mainly produced backwards, as it is clearly indicated by the experimental curves. This implies of course opposite signs for the amplitudes of the S- and P-waves.

We remark that the curves in Fig. 2 seem to show a decrease in the relative weight of the P-waves with respect to the S-waves with increasing energy.

5. – Explicit calculation of the angular correlation.

We give here a detailed calculation for the angular correlation function (5′). following the discussion of Sect. 3.

I) Sharp parity for Λ^0-θ^0; parity conservation in decay,

$$(16)\qquad W(\omega;\vartheta,\varphi) = \begin{cases} \lambda_{ll'} I_+(\omega)\,, & \text{case } a) \\ \lambda_{ll'} I_-(\omega)\,, & \text{case } b) \end{cases}$$

where $\lambda_{ll'}$ indicates λ_{00} or λ_{11} dependently on the choice of the intrinsic parity of the Λ^0, which here is assumed to have a well-definite parity.

It can be easily seen that the angular distribution of the decay shows complete isotropy in ϑ and φ.

II) Under the same assumptions as in I) but no conservation of parity in the decay of the Λ^0 one has:

$$(17)\qquad W(\omega;\,\vartheta,\varphi)=\begin{cases}(\lambda_{00}+\lambda_{11})I_+(\omega)+(\lambda_{01}+\lambda_{10})P_+(\omega)\sin\vartheta\sin\varphi & \text{case } a)\\[2ex] (\lambda_{00}+\lambda_{11})I_-(\omega)+(\lambda_{01}+\lambda_{10})P_-(\omega)\sin\vartheta\sin\varphi & \text{case } b),\end{cases}$$

where

$$(17')\quad P_+(\omega)=(\{101,\,110\}+\{312,\,101\})\,2\sin\omega+\{312,\,110\}\,6\sin\omega\cos\omega \qquad (l_1+l_2=\text{odd}),$$

$$(17'')\quad P_-(\omega)=(\{100,\,111\}+\{311,\,100\})\,2\sin\omega+\{311,\,111\}\,6\sin\omega\cos\omega \qquad (l_1+l_2=\text{even})$$

and where

$$\{2Jl_1l_2,\,2J'l_1'l_2'\}=-i\{a(Jl_1l_2)a^*(J'l_1'l_2')-a^*(Jl_1l_2)a(J'l_1'l_2')\}\,.$$

Here the angular distribution of the decay shows the $\sin\vartheta\sin\varphi$ dependence which means actually up-and-down asymmetry with respect to the plane of production.

III) Parity doublets; parity conservation is the decay of the

$$(18)\qquad W(\omega;\,\vartheta,\varphi)=(\lambda_{00}+\lambda_{11})\,[I_+(\omega)+I_-(\omega)]+ \\ +(\lambda_{01}+\lambda_{10})\,[D_0(\omega)\cos\vartheta+D_1(\omega)\sin\vartheta\cos\varphi].$$

Here the contribution to $\cos\vartheta$ and to $\sin\vartheta\cos\varphi$ are due to interference of doublet states: such a dependence shows an asymmetry in the plane of production left-or-right with respect to the line of flight of the Λ^0. Here one has

$$(18')\quad \begin{cases} D_0(\omega)=2\{[101,\,111]+[100,\,110]-[100,312]-[101,311]\}+ \\ \qquad +2\cos\omega\,\{[100,\,101]+[110,\,111]+2[110,\,311]+2\,[111,\,312]+ \\ \qquad +\frac{2}{5}\,[311,\,312]\}+6\cos^2\omega\{[100,\,312]+[101,\,311]\}\,, \\[2ex] D_1(\omega)=-2\,\{[110,\,111]-[100,\,101]-[110,\,311]-[111,\,312]- \\ \qquad -\frac{4}{5}\,[311,\,312]\}\sin\omega+6\sin\omega\cos\omega\{[311,\,101]+[312,\,100]\}\,. \end{cases}$$

IV) Combining both parity doublets and non-conservation of parity in the decay of the Λ^0, one has the additional effect of both anisotropies, namely:

$$W(\omega;\,\vartheta,\varphi) = (\lambda_{00}+\lambda_{11})[I_+(\omega)+I_-(\omega)]+ \tag{19}$$

$$+(\lambda_{01}+\lambda_{10})\{D_0(\omega)\cos\vartheta + D_1(\omega)\sin\vartheta\cos\varphi + [P_+(\omega)+P_-(\omega)]\sin\vartheta\sin\varphi\},$$

where

$$[P_+(\omega)+P_-(\omega)]\sin\vartheta\sin\varphi \tag{19'}$$

is the anisotropic term arising for non-conservation of parity for doublets.

6. – Final remarks and conclusions.

Before we discuss our results and before we compare them to the experimental curves obtained in Bologna and Pisa, we find it useful to remark that the general features of our calculations would not be changed if the spin S of the Λ^0 particle had a higher value, for instance $S=\frac{3}{2}$. The only difference in the quantitative results would be in the highest orders, L_2 and L of the spherical harmonics involved in the angular distribution. The general structure of it would however not change, the argument given in Sect. **3** being independent on the value of S as far as the sign $(-)^\alpha$ is concerned.

The results so far obtained show that if the decay takes place with conservation of parity and if the parity of the Λ^0 is well-defined, the angular correlation function reduces to the angular distribution for production.

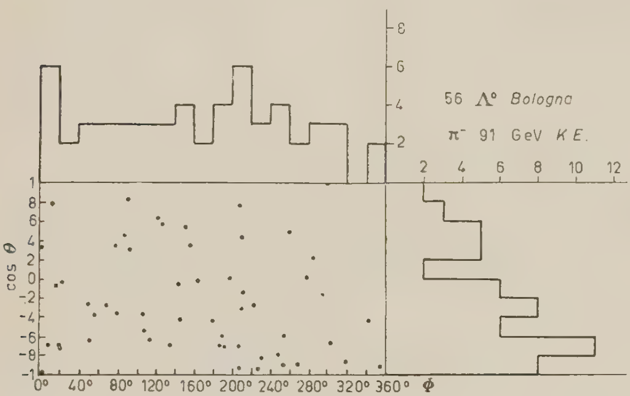

Fig. 6*a*. – Experimental curves for angular correlation at 0.91 GeV.

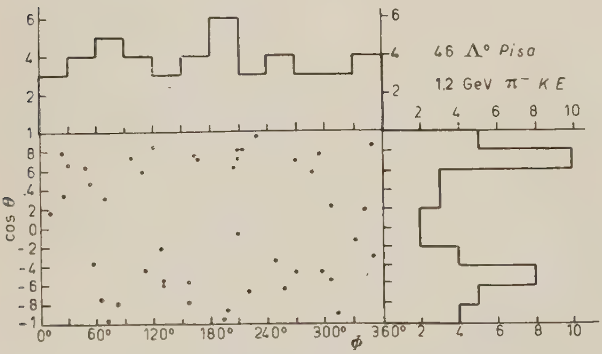

Fig. 6*b*. – Experimental curves for angular correlation at 1.2 GeV.

This implies no dependence on the angle between the planes of production and decay.

Since there is experimental evidence of a dependence of the angular correlation on the angle φ between plane of production and plane of decay, we exclude that the present case be of actual interest.

The experimental curves (Fig. 6 *a*) *b*)) show indeed that there is an evident dependence on the angle between the outgoing π^- and the perpendicular to the plane of production, which is indicated with Θ. The same curves also show a dependence on the angle Φ between the component of the outgoing π^- on the plane of production and the Λ^0. We here point out that both angles are related to the angles ϑ, φ of our frame of reference as follows:

$$\cos\Theta = \sin\vartheta\sin\varphi\,, \qquad \sin\Theta\cos\Phi = \cos\vartheta\,, \qquad \sin\Theta\sin\Phi = \sin\vartheta\cos\varphi\,.$$

The curves in Fig. 6 *a*) and Fig. 6 *b*) show a dependence on $\cos\Theta$ which is remarkably indicative of the non-conservation of parity in the decay. More precisely, the $\cos\Theta$ dependence is actually the same anisotropic dependence which we have pointed out in Sect. **5**, which occurs as a pure effect of non-conservation of parity in decay. From the experimental curves however it would be hard to investigate the value S of the spin of the Λ^0, because there is still too much bias in the curves in order to draw conclusions on the actual order L of the spherical harmonics involved in the angular distribution.

The angular dependence on Φ, as shown by the same curves (Fig. 6 *a*), Fig. 6 *b*)) does not suggest a precise indication of the existence of parity doublets: the distribution in the angle in fact contains a superposition of both contributions:

$$\sin\Theta\cos\Phi\,, \qquad \sin\Theta\sin\Phi\,,$$

arising from parity doublets.

If it were possible to obtain from the statistics of the events the distribution of the events after averaging on the angle φ between the plane of production and the plane of decay, one could check the existence of the $\cos\vartheta$ dependence which is characteristic of parity doublets.

So far, therefore, we cannot decide about the parity doublets, while the effect of parity non-conservation in decay seems to be recognizable.

* * *

We are indebted to Professor G. P. PUPPI for stimulating discussions and very indebted to the experimental group in Bologna for communication of the experimental results.

RIASSUNTO

Si è preso in esame il processo $p+\pi^- \to \Lambda^0+\theta^0$ e il successivo decadimento della Λ^0 nell'intento di ottenere informazioni intorno allo spin e alla parità della particella Λ^0; per questo si sono date espressioni esplicite per le correalzioni angolari usando il metodo della matrice densità. Per quanto riguarda il solo processo di produzione si sono prese in considerazione onde S e P per il sistema Λ^0-θ^0 ottenendo curve che possono trovarsi in accordo con i risultati sperimentali. Nello studio delle correlazioni angolari si sono discussi i casi in cui la particella Λ^0 abbia parità ben definita o sia prodotta invece in uno stato di doppietto di parità e, nei due casi, decada con conservazione o meno di parità. La parità si suppone sempre conservata in produzione. Nel caso di non conservazione di parità nel decadimento si trova una dipendenza secondo $\sin\vartheta \sin\varphi$, essendo ϑ l'angolo tra il π^- di decadimento e la Λ^0 e φ l'angolo tra il piano di produzione e il piano di decadimento. Questa anisotropia sembra confermata dai risultati sperimentali di Bologna e di Pisa. Nel caso di doppietti di parità si trova una dipendenza secondo il $\cos\vartheta$ e secondo $\sin\vartheta\cos\varphi$. Il confronto con l'esperienza potrebbe farsi dall'analisi delle distribuzioni angolari in ϑ dopo aver mediato su φ; una eventuale anisotropia in $\cos\vartheta$ confermerebbe l'ipotesi dei doppietti. I risultati sperimentali finora ottenuti non ci permettono di trarre conclusioni sicure al riguardo.

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Remarks on the Measurability of Electromagnetic Fields.

G. Heber

Joint Institute for Nuclear Research - Dubna, USSR (*)

(ricevuto il 14 Novembre 1957)

Summary. — Indeterminacy relations are derived, which express the facts, that only 4-dimensional averages of electromagnetic fields are measurable and that also an average cannot be measured with unlimited accuracy simultaneously with the position of the averaging region in space-time continuum. Essentially in showing this is the use of test bodies, which obey Heisenbergs indeterminacy relations and possess a finite extension, finite mass and finite electric charge.

1. - Introduction.

As is well known there are growing in last time doubts about the consistency of conventional, local quantum field theories and also concrete difficulties inside these theories. It is quite clear, that the grave difficulties come all from very big momenta or form the smallest space-time regions. It is therefore accepted by most physicists, that the field theory should be alterated anyhow in the smallest space-time regions. But up to date it seems that, in spite of numerous trials in many different directions (+), a satisfactory way out of this situation has not yet been found. Looking at this situation it might be right to discuss the question, how far the fundamental principles of local field are fit to describe nature. It is hoped to get from such a consideration some hint from known properties of nature to a likely still unknown way, leading us to a better field theory. In another formulation our question is:

(*) On leave of absence from the University of Jena; permanent address: Institute of Theoretical Physics, University of Jena, Germany.

(+) Compare, e.g. the review-article by Blohincev [1].

[1] D. Blohincev: *Usp. Fiz. Nauk*, **61**, 137 (1957).

How deep are the fundamental principles of local field founded in reality, or even: Is it possible to measure in reality the fundamental quantities of conventional field theory exactly? (*).

We shall restrict ourselves in this work to the best-known field theory, electrodynamics. Our concrete question is: How far are the field quantities of electrodynamics (in classical and in quantum theory) accessible to real measurements?

Looking hastily at this problem one could think it completely settled by the paper of BOHR and ROSENFELD (2). These authors indeed have shown (by means of thought-experiments), that it is possible to measure the average of any component of the electromagnetic field over a 4-dimensional region with unlimited accuracy. It seems that in (2) the extension of the region of averaging could get as small as you wish. But it is an essential supposition of this limiting process, as stressed by the authors themselves, that the atomic structure of the measuring-device (especially of the test-bodies) can be neglected. From a *logical* point of view it is possible to make such supposition, because electrodynamics (without coupling to any other field) is free of any hint to atomic structure of matter. But of course for *practical* measurements the atomic structure of matter must be taken into account, if one wants to measure in space-time dimensions of atomic size e.g.

In the followimg sections it will be shown by very simple arguments that it is indeed impossible to determine simultaneously the average of one component of the electromagnetic field over a microscopical space-time region and the position of this region exactly by means of really existing test-bodies. Indeterminacy relations for this phenomenon are derived.

2. - Measurement of electric fields.

At first we shall shortly summarize those points, which in the proof of BOHR and ROSENFELD (2) prevent its extension to very small space-time regions, taking into account the atomic structure of matter. Once more we wish to stress, that of course BOHR and ROSENFELD were quite aware of these points, as can be seen from (2) and still more clear from a more recent article of ROSENFELD (3). The most critical propositions of (2) are:

(*) We return to the old principle, which in the hands of Heisenberg was very fertile in establishing quantum mechanics that it should be *possible* to formulate a physical theory by no other than measurable quantities.

(2) N. BOHR and L. ROSENFELD: *Kgl. Danske Vid. Selsk. Math.-fyz. Medd.*, **12**, No. 8 (1933).

(3) L. ROSENFELD: in *Niels Bohr and the Development of Physics* (London, 1955), esp. pp. 83, 84. Look also: JAUCH and ROHRLICH: *Theory of Photons and Electrons* (Cambridge, 1955), p. 81.

a) Mass and electric charge of the test-body can be brought to infinity (*).

b) One should be able to cut the test-body into many small parts, each of them wearing a mirror.

c) Before and after measurement the test-body should be bound at the same point of a lattice, fixed in space.

d) During measurement the body should be connected with the just mentioned lattice by a linear spring.

It is clear, that such test-body, if in reality it is existing at all (+), must have macroscopic dimensions. Of course you can think of a body with such qualities, having any size; therefore these propositions are *logically* correct. But for our problem we have to take as test-bodies for the smallest space-time regions only such bodies, which are found *really* in nature. Not useful for our purpose are also particles with a point-structure or bodies (particles) with a not well-defined (×) or with a not well-known structure. The first type of particles should be excluded because there are quantum fluctuations of infinite size acting on them (to be derived from conventional theory, look e.g. CORINALDESI ([5]); about measurements with point-particles see however the paper by LANDAU and PEIERLS ([6]). The second type of particles is not useful because the space-structure of electric charge of the text-body determines to an essential part the space-region in which the field should be averaged; this structure therefore should be known as well as possible, it should be most simple and very stable. As we wish to measure in space-time regions, being as small as possible, it seems to the author, that atomic nuclei are now the most useful test-bodies (†). The following discussion however is not only valid for atomic nuclei but also for any other test-bodies, fulfilling the above mentioned propositions.

The method of measuring the electric field by such test-bodies we take from BOHR and ROSENFELD ([2]): basically is the formula:

$$\overline{\mathfrak{E}}\,\varrho V T = \mathfrak{p}_2 - \mathfrak{p}_1 \,. \tag{1}$$

The meanings of the symbols in (1) are:

ϱ: Density of electric charge of the test-body (assumed to be constant);

(*) This point was critized already in the early work of MARKOV ([4]).

(+) I.e., if it is built up from matter, which is taken from the known part of our world.

(×) Here we mean also: Too much complicated.

(†) This also was the opinion of MARKOV ([4]).

([4]) M. MARKOV: *Phys. Zeits. Sowj.*, **12**, 105 (1937); see also, *Žu. Ėksper. Teor. Fiz.*, **8**, 124 (1938).

([5]) E. CORINALDESI: *Suppl. Nuovo Cimento*, **10**, 83 (1953).

([6]) L. LANDAU and R. PEIERLS: *Zeits. f. Phys.*, **69**, 56 (1931).

V: Volume of the test-body and of the averaging-region in space;

$\bar{\mathfrak{E}} = (1/VT) \int_{V,T} \mathfrak{E}(r, t)\, d^3r\, dt$: Average of electric field $\mathfrak{E}$ over the 4-dimensional region V, T (in quantum theory: Eigenvalue of the average...);

T: Interval of time, giving duration of measuring and averaging interval in time; T could be taken arbitrary inside certain limits;

$\mathfrak{p}_1$, $\mathfrak{p}_2$: Momenta of the test-body at the beginning and end of T.

If ϱ, V, T are known, we can take by (1) from a measurements of $\mathfrak{p}_1 - \mathfrak{p}_2$ of course $\bar{\mathfrak{E}}$. But by measuring exactly the moment of the test-body, you make, according to $\Delta p_\xi \Delta \xi \geqslant \hbar$ ..., ..., the position (here the position ξ of the centre) of the body uncertain. Also you need for an exact determination of the momentum according to quantum mechanics an infinite time. That means: By measuring exactly $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the position of the averaging region V, T in the space-time-continuum gets completely indefinite. Of course an exact determination of $\bar{\mathfrak{E}}$ has no sense if it is not possible to know, where and when this field was existing. Therefore it is relevant to make both measurements of momentum only with an inaccuracy Δp_ξ chosen such, that $\Delta\xi$ gets one order of magnitude smaller than the diameter of the test-body (*). A rough estimate shows, that the resulting indeterminacy $\Delta\tau$ of T is then extremely small (for an atomic nucleus $\Delta\tau \approx 10^{-25}$ s).

The need for a certain localization of the process of measuring the field in space-time compells us therefore to let the field average itself uncertain in the order of:

$$\Delta\bar{\mathfrak{E}}_\xi \approx \frac{\Delta p_\xi}{\varrho V T}, \tag{2}$$

or also

$$\Delta\bar{\mathfrak{E}}_\xi > \frac{\hbar}{\varrho V T \Delta\xi}. \tag{3}$$

The inequality (3) of course is contained already in the paper by Bohr and Rosenfeld [2]. The essential difference being, that in [2] ϱ is allowed to have any value. Therefore by letting $\varrho \to \infty$, the right side of (3) can be brought to zero. Our proposition of taking only really existing particles as test-bodies clearly limits the variability of ϱ. Indeed the sort of matter with the highest ϱ we know now is nuclear matter. There seems to be very little probability of

(*) Of course we can confine our discussion to the determination of one component of $\bar{\mathfrak{E}}$, say $\bar{\mathfrak{E}}_\xi$.

finding any other sort of matter with a well-defined ϱ, being still more dense. In the authors opinion therefore (3) represents an essential limit of measurability of the electric field in reality.

Now we remark still two points, which were neglected in discussing (3): Firstly the averaging-volume V, used to define $\bar{\mathfrak{E}}$, is not constant in time, according to the action of the field and because mostly we have $\mathfrak{p}_1 \neq 0$. This is not convenient. Therefore one should manage, that in measuring the electric field one has $\mathfrak{p}_1 \approx 0$. The problem gets most simple also if the field to be measured is very weak and if the test-body is as heavy as possible. It may be, that for the measurement of stronger fields any sort of compensation-device will proof convenient.

Secondly we would like to correct the measured field $\bar{\mathfrak{E}}$ for the own-field of the test-body. Of course this is possible as soon as we know the world-line of the test-body in the past and as soon as classical electrodynamics are competent. In our problem however neither classical electrodynamics are exactly competent nor is it possible to know exact data on the past of our test-body. It seems therefore to the author, that it is impossible to perform this correction; we should consider the own-field of a microscopic test-body as inseparable part of the measured field.

3. - Measurement of magnetic fields.

In macro-physics it is possible to measure a magnetic field by the force, with which it acts on an electric current or on a magnet. In micro-physics one has to remember, that a current consists of single charged particles and it seems to be most simple to use for measurement the power, by which a magnetic field $\mathfrak{B}$ acts on a single, moving, charged body (velocity $\mathfrak{v}$, charge density ϱ) (*). The density of this power is, as is well-known,

$$\mathfrak{k} = \varrho\,\frac{\mathfrak{v}\times\mathfrak{B}}{c} = \frac{\varrho}{c}\left\{\begin{matrix} \mathfrak{v}_y\mathfrak{B}_z - \mathfrak{v}_z\mathfrak{B}_y \\ \mathfrak{v}_z\mathfrak{B}_x - \mathfrak{v}_x\mathfrak{B}_z \\ \mathfrak{v}_x\mathfrak{B}_y - \mathfrak{v}_y\mathfrak{B}_x \end{matrix}\right\}. \tag{4}$$

For the measurement of a magnetic field of the special form $\mathfrak{B} = (0, 0, \mathfrak{B}_z)$, one has therefore two simple possibilities: Taking either a particle of velocity $\mathfrak{v} = (0, v_y, 0)$ or $\mathfrak{v} = (v_x, 0, 0)$. We should take $|\mathfrak{v}|$ not too small, in order to

(*) Taking the power on a magnet we need a new test-body: we can take however the same test-body as in Sect. **2** by using (4).

make $\mathfrak{B}$ strong and well measurable. Of course also here we can measure only an average-value of the field over a finite space-time-region V, T; the fundamental equation for the second possibility reads:

$$\bar{\mathfrak{B}}_\zeta \varrho V T \frac{v_\xi}{c} = -(p_{\eta_1} - p_{\eta_2})\,. \tag{5}$$

The symbols in (5) are defined quite parallel to (1). Also the further considerations are quite analogous to those of Sect. 2. Only v_ξ in (5) must be considered separately. In order to know something about the position ξ of the test-body in the ξ-direction, v_ξ cannot be known exactly. It is however convenient and possible to choose $p_\xi \ll \Delta p_\xi$; Δv_ξ has then practically not any influence on $\Delta\bar{\mathfrak{B}}_\xi$. Under this proposition we have:

$$\Delta\bar{\mathfrak{B}}_\zeta \approx \frac{\Delta p_\eta \cdot c}{\varrho V T v_\xi} \quad \text{or also} \quad \Delta\bar{\mathfrak{B}}_\zeta \geqslant \frac{\hbar c}{\varrho V T v_\xi \Delta\eta}\,. \tag{6}$$

Of course there are still 5 other relations for the uncertainties in measuring magnetic fields, each being quite analogous to (6). From (6) it follows quite similar to the discussion of (3), that it is impossible to determine simultaneously the average of the magnetic field with the position of the averaging-region. Also it is impossible to take the limit of V, $T \to 0$, without getting an infinite indeterminacy in the field-measurement.

4. – 4-dimensional formulation.

The inequalities (3), (6) and the 7 other analogous relations can be understood very clearly, using the notation of the theory of relativity. Introducing:

$$j_\nu = \varrho(c, v_\xi, \mathfrak{v}_\eta, v_\xi); \qquad \xi_\nu = (c\tau, \xi, \eta, \zeta) \ \ (*); \qquad \Delta\xi_\nu \text{ analogous};$$

$$f_{\mu\nu} = \begin{pmatrix} 0 & \mathfrak{E}_x & \mathfrak{E}_y & \mathfrak{E}_z \\ -\mathfrak{E}_x & 0 & \mathfrak{B}_z & -\mathfrak{B}_y \\ -\mathfrak{E}_y & -\mathfrak{B}_z & -0 & \mathfrak{B}_x \\ -\mathfrak{E}_z & \mathfrak{B}_y & -\mathfrak{B}_x & 0 \end{pmatrix}; \qquad \bar{f}_{\mu\nu}(\xi) = \frac{1}{VcT}\int\limits_{V,cT(\xi)} f_{\mu\nu}(x)\,\mathrm{d}^4x\,;$$

$\Delta f_{\mu\nu}$ analogous, it is possible to comprehend the mentioned 9 inequalities to:

$$\Delta\bar{f}_{\mu\nu}(\xi)\,\Delta\xi_\lambda\, j_K \geqslant \frac{\hbar c}{VT}\,\delta_{\mu K,\nu\lambda}\,, \tag{7}$$

(*) By ξ_ν we symbolize the co-ordinates of the centre of V, T.

the tensor δ having the following meaning:

$$\begin{aligned} &\delta_{00,ii} = 1 = -\,\delta_{ii,00} && \text{for } i = 1, 2, 3 \\ &\delta_{ii,KK} = 1 = -\,\delta_{KK,ii} && \text{for } i \neq K;\ i, K = 1, 2, 3 \\ &\delta_{\mu K,\nu\lambda} = 0 && \text{else.} \end{aligned}$$

As it should be, also the other 3 components of (7), which do not belong to the above mentioned 9 inequalities, have physical meaning. One of these 3 relations is:

$$\text{(8)} \qquad \Delta\bar{f}_{01}(\xi)\,\Delta\xi_0 j_1 \geqslant -\frac{\hbar c}{VT}, \qquad \text{or} \qquad \Delta\bar{\mathfrak{E}}_\xi(\xi)\,\Delta\tau\, v_\xi \geqslant \frac{\hbar}{\varrho VT}.$$

According to (2) we can give (8) also the form $\Delta p_\xi \Delta\tau\, v_\xi \geqslant \hbar$: using $\Delta p_\xi v_\xi \approx \Delta E$, we get instead of (8) $\Delta E\,\Delta\tau \geqslant \hbar$. Here ΔE means the uncertainty of the energy of our test-body, $\Delta\tau$ is the indeterminacy of time of measurement. (8) represents therefore the well-known relation between energy and time.

5. – Concluding remarks.

At first still some remarks on the validity of the above estimates:

Everywhere we did assume, that the test-body during all the measuring-process does not suffer any deformation or other alteration (of course, translations are allowed). This is only valid for weak fields and for not too small $\Delta\xi$ (*).

It might further be remarkable, that the relations (7) are valid in classical *and* in quantized electrodynamics, as soon as the test-bodies are particles, which obey quantum-mechanics.

Of course it must be stressed also, that our discussion does not exclude the possibility of the existence of any other methods of measuring electromagnetic fields (also indirect methods), being more exact than the method analysed by us. The author however does not give a big probability to this possibility.

Concerning the meaning of the inequalities (7), one can see already the following. Clearly they are not in accord with the conventional, local electrodynamics. They withdraw the direct empirical foundation of that theory by saying: The field-quantities of Maxwells theory at space-time-points are not measurable. Only the averages of the fields over finite space-time regions are measurable. But even the averages cannot be determined quite exact, if one wishes to measure simultaneously the position of the averaging-region in

(*) For very small $\Delta\xi$ namely the process of measuring the position ξ shall become a strong perturbation, by which the test-body and the field might be altered essentially and statistically.

space-time continuum. Therefore we can have no doubt, that one has to try to reformulate electrodynamics essentially, if one wishes to eliminate all non-observable quantities. It shall be most important to limit anyhow the concept of the local field. It might be typical for this limitation, that certain qualities of the elementary test-bodies shall be contained organically inside the fundamental equations of the field (*). The up-to-date usual sharp separation between field and test-body must likely be given up. May be, this can be interpreted as a new example for the general trend of modern physics to take into account the essential statistical interaction between measuring-apparatus and measuring-object.

It is quite clear that this new theory should contain also the conventional theory as a limiting case for big averaging-regions. It should contain especially all the successes of the conventional theory.

Of course there is also the possibility that the just sketched program cannot be fulfilled. In that case the author would vote for searching a theory without any field-quantities.

Concluding we wish to mention a deep-lying difficulty, appearing in the process of measuring more than one field-average simultaneously by atomic (or nuclear) test-bodies (+). Surely the natural way would be to take identical test-bodies. But this seems to be impossible, because identical atomic objects cannot be distinguished. At the end of the measurement one could not know, how to associate the bodies to those of the beginning of measurement. Author does not yet see any way out of this difficulty.

* * *

It is a pleasure to the author to thank the direction of the Joint Institute for Nuclear Research for the kind hospitality. To many members of the staff of this Institute the author ows thanks for interesting discussions, especially to Professors MARKOV, BLOHINCEV, VOTRUBA and BOGOLJUBOV.

(*) A similar opinion was put forward by MARKOV (4).

(+) Discussion of such processes leads in (2) to the conventional uncertainty relations between two field-components.

RIASSUNTO (*)

Si derivano relazioni di indeterminazione esprimenti il fatto che solo medie quadridimensionali di campi elettromagnetici sono misurabili e che anche una media non può esser misurata con esattezza illimitata simultaneamente alla posizione della regione sulla quale si esegue la media nel continuo spazio-temporale. Per la dimostrazione di quanto sopra è essenziale servirsi di corpi di prova obbedienti alle relazioni di indeterminazione di Heisenberg ed aventi estensione, massa e carica elettrica finite.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Modes in Rectangular Guides Filled with Magnetized Ferrite (*).

G. Barzilai

Istituto Superiore Poste e Telecomunicazioni, Fondazione « U. Bordoni »
Istituto Elettrotecnico dell'Università - Roma

G. Gerosa

Istituto Elettrotecnico dell'Università - Roma

(ricevuto il 18 Novembre 1957)

Summary. — A general modal solution for a rectangular guide, with a slab of ferrite against one side wall and magnetized in a direction parallel to this wall and normal to the axis of the guide, is found and the characteristic equation is obtained. By specializing this equation for the case of the guide completely filled, the relative characteristic equation is derived. A detailed discussion of this last equation, together with an analysis of the possible field configurations, is given. Numerical calculations are carried out in order to clarify the modes distribution. The preceding analysis is finally used to derive the modes cut-off dimensions for the completely filled guide.

1. – Introduction.

It is the purpose of this paper to derive some theoretical results relative to the propagation of electromagnetic waves in rectangular guides loaded with magnetized ferrite.

We shall first consider a guide with a slab of ferrite against one side wall and magnetized in a direction parallel to this wall and normal to the axis of the guide. Kales and co-authors [1], Lax and co-authors [2,3] and others have

(*) Presented orally at the *Congrès International Circuits et Antennes Hyperfréquences*, Paris, 21-26 October 1957.

(1) M. L. Kales, H. N. Chait and N. G. Sakiotis: *Journ. Appl. Phys.*, **24**, 816 (1953).

(2) B. Lax, K. J. Button and L. M. Roth; *Journ. Appl. Phys.*, **25**, 1413 (1954).

(3) K. J. Button and B. Lax: *Trans. IRE*, AP-**4**, 531 (1956).

solved and discussed this problem assuming a solution with no dependence in the direction of the d.c. magnetic field. To our knowledge, however, no one seems to have worked out a general solution of this problem; we have therefore thought of interest to do this and to derive the relative characteristic equation.

We want to briefly mention here that this and similar problems (4), and even those more complicated, like the ones relative to guides with one or more slabs of ferrite away from the side walls, do not present particular difficulty in their setting up, since a straightforward approach is quite adequate. The difficulties lie in the algebraic manipulations necessary to put the solution in a manageable form and in the discussion and the interpretation of the results.

By specializing our solution, we shall start by discussing the case of the guide completely filled with transversely magnetized ferrite.

The solution for this case was originally given by MIKAELIAN (5), but the only discussion available up to now, with no restriction imposed on the field dependences, seems to be the one given by SEIDEL (6) for the asymptotic case of guides of vanishingly small cross-section. We have therefore thought it to be of interest to give a detailed discussion of the modes spectrum for guides of finite cross-section. A detailed discussion of the general solution found will be the object of some future work.

2. – The fields in unbounded regions.

To construct our general solution we need the expressions of a general field in unbounded regions filled with magnetized ferrite and with isotropic medium.

Let us consider first an unbounded region filled with uniformely magnetized ferrite. Refer this region to a right hand system of rectangular co-ordinates x, y, z. Assume that the d.c. magnetic field, of sufficient intensity H_0 to saturate the ferrite, is directed along the z axis.

It is well known that the dielectric constant of such a medium is a scalar $\varepsilon_0 \varepsilon$ and the magnetic permeability is a tensor $\boldsymbol{\mu}$ given by the following expression:

$$\boldsymbol{\mu} = \mu_0 \begin{vmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{1}$$

(4) A. L. MIKAELIAN: *Radioteknika*, **10**, n. 9, p. 3 (1955).
(5) A. L. MIKAELIAN: *Dokl. Akad. Nauk SSSR*, **98**, 941 (1954).
(6) H. SEIDEL: *Bell System Tech. Journ.*, **36**, 409 (1957).

where

$$\mu_1 = 1 + \frac{\varrho}{1-\tau^2}\,, \qquad \mu_2 = \frac{\tau\varrho}{1-\tau^2}\,,$$

$$\varrho = \frac{M_0}{\mu_0 H_0}\,, \qquad \tau = \frac{\omega}{\omega_0}\,,$$

and M_0 is the intensity of the saturation magnetization, ω and $\omega_0 = -\gamma H_0$ are the applied and the resonant circular frequencies, γ is the gyromagnetic ratio for the electron, μ_0 and ε_0 are the permeability and the dielectric constant of the vacuum.

Assuming $\exp[j\omega t]$ as time dependence, consider the following field:

$$\begin{Bmatrix} \boldsymbol{E}_f \\ \\ \boldsymbol{H}_f \end{Bmatrix} = \begin{Bmatrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ \\ 1 \end{Bmatrix} A \begin{Bmatrix} \boldsymbol{e}_f \\ \\ \boldsymbol{h}_f \end{Bmatrix} \exp\left[j(k_{xf}x + k_{yf}y + k_{zf}z)\right]\,, \tag{2}$$

where: k_{xf}, k_{yf} and k_{zf} are the propagation constants measured in units ot $\omega\sqrt{\mu_0\varepsilon_0}$ and consequently lengths are measured assuming as unif $1/\omega\sqrt{\mu_0\varepsilon_0}$;

$\boldsymbol{e}_f$ and $\boldsymbol{h}_f$ are two constant adimensional vectors;

A is an amplitude factor.

By introducing (2) into Maxwell's equations we obtain:

$$\mu_1 t^4 + [(\mu_1+1)k_{zf}^2 - \varepsilon(\mu_1^2 - \mu_2^2 + \mu_1)]t^2 + k_{zf}^4 - 2\mu_1\varepsilon k_{zf}^2 + \varepsilon^2(\mu_1^2 - \mu_2^2) = 0\,, \tag{3}$$

$$\left|\begin{aligned}
e_{xf} &= -k_{zf}[(\mu_1 - 1)k_{xf}k_{yf} + j\mu_2(\varepsilon - k_{xf}^2)] \\
e_{yf} &= -k_{zf}[k_{zf}^2 - \mu_1(\varepsilon - t^2) - (\mu_1 - 1)k_{xf}^2 - j\mu_2 k_{xf}k_{yf}] \\
e_{zf} &= j\mu_2 k_{xf}(\varepsilon - t^2) + k_{yf}[k_{zf}^2 - \mu_1(\varepsilon - t^2)] \\
h_{xf} &= k_{xf}^2[k_{zf}^2 - (\varepsilon - t^2)] - \varepsilon[k_{zf}^2 - \mu_1(\varepsilon - t^2)] \\
h_{yf} &= k_{xf}k_{yf}[k_{zf}^2 - (\varepsilon - t^2)] + j\mu_2\varepsilon(\varepsilon - t^2) \\
h_{zf} &= -k_{zf}[k_{xf}(\mu_1\varepsilon - k_{zf}^2 - t^2) + j\mu_2\varepsilon k_{yf}]
\end{aligned}\right. \tag{4}$$

where:

$$t^2 = k_{xf}^2 + k_{yf}^2\,. \tag{5}$$

Equation (3) is the characteristic equation for an unbounded region filled with magnetized ferrite.

We shall write now the corresponding solution for free space (*), cast in a form suitable for our purposes.

$$
\left\{ \begin{matrix} \boldsymbol{E}_0 \\ \\ \boldsymbol{H}_0 \end{matrix} \right\} = \left\{ \begin{matrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ \\ 1 \end{matrix} \right\} \left[B_1 \left\{ \begin{matrix} \boldsymbol{e}_{01} \\ \\ \boldsymbol{h}_{01} \end{matrix} \right\} + B_2 \left\{ \begin{matrix} \boldsymbol{e}_{02} \\ \\ \boldsymbol{h}_{02} \end{matrix} \right\} \right] \exp\left[j(k_{x0}x + k_{y0}y + k_{z0}z) \right], \tag{6}
$$

where the propagation constants and lengths have been normalized in the same way as before, B_1 and B_2 are two arbitrary amplitude factors, and:

$$
\left\{ \begin{array}{llll}
e_{x01} = -k_{z0}(1-k_{x0}^2) & & e_{x02} = 0 \\
e_{y01} = k_{x0}k_{y0}k_{z0} & & e_{y02} = k_{z0} \\
e_{z01} = k_{x0}k_{z0}^2 & & e_{z02} = -k_{y0} \\
h_{x01} = 0 & & h_{x02} = 1-k_{x0}^2 \\
h_{y01} = k_{z0}^2 & & h_{y02} = -k_{x0}k_{y0} \\
h_{z01} = -k_{y0}k_{z0} & & h_{z02} = -k_{x0}k_{z0}
\end{array} \right. \tag{7}
$$

$$
k_{x0}^2 + k_{y0}^2 + k_{z0}^2 = 1\,. \tag{8}
$$

3. - The modes for the parallel plate guide partially filled with ferrite.

Let us refer to Fig.1.

We shall assume the walls to be perfectly conducting.

Each modal solution is labelled by a pair of values k_z, k_x. In order to satisfy the boundary conditions at the interface we must have: $k_{zf} = k_{z0} = k_z$ and $k_{xf} = k_{x0} = k_x$.

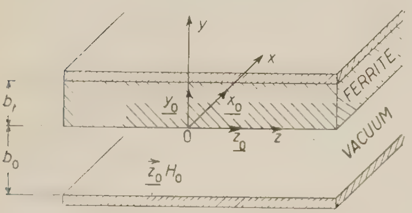

Fig. 1. – Geometry of the parallel plate guide partially filled with ferrite.

For the ferrite region equation (3) yields, for a given k_z, two values of t^2: t_1^2 and t_2^2. From equation (5) we obtain two values of k_{yf}^2: k_{y1}^2 and k_{y2}^2, for a given k_x. We can conclude that to each pair k_z, k_x there correspond four solutions of the form (2).

We shall therefore express the field in the ferrite, to construct our modal solution, as a superposition of four solutions corresponding to a given pair k_z, k_x.

(*) If instead of vacuum the guide contains an isotropic dielectric of absolute dielectric constant ε_d, what follows will still be valid provided in all the formulae ε_0 is replaced with ε_d.

Operating in this way we obtain:

$$(9)\qquad \begin{Bmatrix} \boldsymbol{E}_f \\ \boldsymbol{H}_f \end{Bmatrix} = \begin{Bmatrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ 1 \end{Bmatrix} \Bigg[A_1^+ \begin{Bmatrix} \boldsymbol{e}_{f1}^+ \\ \boldsymbol{h}_{f1}^+ \end{Bmatrix} \exp[jk_{y1}y] + A_1^- \begin{Bmatrix} \boldsymbol{e}_{f1}^- \\ \boldsymbol{h}_{f1}^- \end{Bmatrix} \exp[-jk_{y1}y] + $$

$$ + A_2^+ \begin{Bmatrix} \boldsymbol{e}_{f2}^+ \\ \boldsymbol{h}_{f2}^+ \end{Bmatrix} \exp[jk_{y2}y] + A_2^- \begin{Bmatrix} \boldsymbol{e}_{f2}^- \\ \boldsymbol{h}_{f2}^- \end{Bmatrix} \exp[-jk_{y2}y] \Bigg] \exp j[(k_x x + k\, z)]\,,$$

where the symbols have obvious significance.

For the vacuum region a general solution corresponding to the same pair k_z, k_x has the form:

$$(10)\qquad \begin{Bmatrix} \boldsymbol{E}_0 \\ \boldsymbol{H}_0 \end{Bmatrix} = \begin{Bmatrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ 1 \end{Bmatrix} \Bigg[B_1^+ \begin{Bmatrix} \boldsymbol{e}_{01}^+ \\ \boldsymbol{h}_{01}^+ \end{Bmatrix} \exp[jk_{y0}y] + B_1^- \begin{Bmatrix} \boldsymbol{e}_{01}^- \\ \boldsymbol{h}_{01}^- \end{Bmatrix} \exp[-jk_{y0}y] + $$

$$ + B_2^+ \begin{Bmatrix} \boldsymbol{e}_{02}^+ \\ \boldsymbol{h}_{02}^+ \end{Bmatrix} \exp[jk_{y0}y] + B_2^- \begin{Bmatrix} \boldsymbol{e}_{02}^- \\ \boldsymbol{h}_{02}^- \end{Bmatrix} \exp[-jk_{y0}y] \Bigg] \exp[j(k_x x + k_z z)]\,.$$

By imposing the tangential component of $\boldsymbol{E}_f$ for $y = b_f$ and of $\boldsymbol{E}_0$ for $y = -b_0$ to be zero and the tangential component of $\boldsymbol{E}_f$ and $\boldsymbol{H}_f$ to be equal to the tangential component of $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ for $y = 0$, we obtain a system of eight linear algebraic equations in the eight unknown amplitudes A and B. By setting equal to zero the determinant of the coefficients, we obtain, after elaborate algebraic manipulations, the following characteristic equation:

$$(11)\qquad k_{y1}k_{y2}C_0 + k_{y1}C_1 \cos k_{y1}b_f \sin k_{y2}b_f + k_{y2}C_2 \sin k_{y1}b_f \cos k_{y2}b_f +$$
$$+ k_{y1}k_{y2}C_3 \cos k_{y1}b_f \cos k_{y2}b_f + C_4 \sin k_{y1}b_f \sin k_{y2}b_f = 0\,,$$

where:

$$\begin{aligned}
C_s &= P_s \sin^2 k_{y0}b_0 + Q_s \cos^2 k_{y0}b_0 + R_s \sin k_{y0}b_0 \cos k_{y0}b_0 \qquad s = 0, 1, 2, 3, 4\\
P_0 &= \varepsilon Q_0 = 2\mu_2^2 \varepsilon k_{y0} k_z^2\\
R_0 &= -2\mu_2 k_x k_z^2[(\varepsilon\mu_1 - 1)(\mu_1 - 1) - \varepsilon\mu_2^2]\\
P_{1,2} &= \mu_2 k_x k_{y0}[(\varepsilon\mu_1 - 1)\, t_{2,1}^2 k_z^2 - q_{1,2}k_z^2 - q_{2,1}(\varepsilon - t_{2,1}^2)]\\
Q_{1,2} &= 0
\end{aligned}$$

$$R_{1,2} = -\frac{k_x^2}{\varepsilon}\left[\mu_2^2 \varepsilon k_z^2 (k_z^2 + \varepsilon + 1) + (\mu_1 - 1) q_{2,1} k_z^2 (k_z^2 - 1 + t_{2,1}^2) + q_{2,1}^2 (k_z^2 + t_{2,1}^2) + \right.$$

$$\left. + \mu_1 q_{1,2}^2 (1 - k_z^2)(k_z^2 - \varepsilon + t_{1,2}^2)\right] + \mu_2^2 k_z^2 (\varepsilon - \varepsilon k_z^2 + t_{2,1}^2) + q_{1,2}^2 (1 - k_z^2) + \frac{q_{2,1}^2 t_{2,1}^2}{\varepsilon},$$

$$P_3 = k_{y0} (q_1^2 + q_2^2)$$

$$Q_3 = - 2\mu_2^2 k_{y0} k_z^2$$

$$R_3 = 2\mu_2 k_x k_z^2 [(\mu_1 - 1)(k_z^2 - 1) + q_1 + q_2]$$

$$P_4 = \mu_2^2 \varepsilon k_{y0} k_z^2 [2k_x^2 - (t_1^2 + t_2^2)]$$

$$Q_4 = -\frac{k_{y0}}{\varepsilon}\left\{\frac{k_x^2}{\varepsilon}[\mu_1 q_2 t_2^2 (k_z^2 - \varepsilon + t_2^2) + \mu_1 q_1 t_1^2 (k_z^2 - \varepsilon + t_1^2) + 2\mu_2^2 \varepsilon k_z^4] - q_1^2 t_1^2 - q_2^2 t_2^2\right\},$$

$$R_4 = \mu_2 \frac{k_x}{\varepsilon}\left\{k_x^2 [\mu_1 (t_1^4 + t_2^4)(k_z^2 - 1) - 2\mu_1 t_1^2 t_2^2 k_z^2 - \mu_1 (t_1^2 + t_2^2)(\varepsilon k_z^2 + k_z^2 - 2\varepsilon) + \right.$$

$$+ 2\mu_1 \varepsilon (k_z^2 - \varepsilon)(k_z^2 - 1) - q_1 k_z^2 (\varepsilon - 1 + t_1^2 - t_2^2) - q_2 k_z^2 (\varepsilon - 1 + t_2^2 - t_1^2)] +$$

$$+ \mu_1 k_z^2 (t_1^2 - t_2^2)^2 + \varepsilon q_1 [k_z^2 (k_z^2 + \varepsilon - 1 + t_1^2) - (\varepsilon - t_1^2)] +$$

$$\left. + \varepsilon q_2 [k_z^2 (k_z^2 + \varepsilon - 1 + t_2^2) - (\varepsilon - t_2^2)]\right\}$$

$$q_{1,2} = k_z^2 - \mu_1 (\varepsilon - t_{1,2}^2) .$$

Explicit expressions for the amplitudes A and B can be obtained by the determinant mentioned above.

Equation (11) constitutes an additional relation between k_z^2 and k_x which limits the arbitrariness of these two quantities.

The modal solution (9)–(10) possesses reflection symmetry along the z direction; this however is not true for the x direction. It can be immediately seen by substituting $-k_z$ in place of $+k_z$ in (9)–(10) and verifying that e_x, e_y, h_z change their sign, while e_z, h_x, h_y do not undergo any change. This characteristic of the solution allows us to find a modal solution for a rectangular guide partially filled with transversely magnetized ferrite, by simply superimposing two solutions of the form (9)–(10) corresponding to equal and opposite values of k_z, but having the same k_x and the same amplitude. In fact by closing the guide of Fig. 1 with two parallel perfectly conducting planes normal to the z axis and situated a distance a apart, the above mentioned modal solution will satisfy the boundary conditions at the surfaces of the new walls, if

$$k_z = \frac{m\pi}{a} . \qquad m = 0, 1, 2, \ldots$$

Since equation (11) contains in general odd powers of k_x, the structure is in general not reciprocal. However in special cases the odd powers of k_x may disappear and in this case the structure becomes reciprocal.

4. – Discussion of the case of the guide completely filled with ferrite.

It can be easily recognized that by suitably specializing the various quantities in equation (11) we obtain the equations given by other authors solving directly particular cases.

For $b_f - 0$, the characteristic equation of the empty guide is immediately obtained.

By letting $k_z = 0$, we obtain the equation given by Kales and co-authors [1].

Finally for $b_0 = 0$ we obtain (by letting $b_f = b$ and normalizing the propagation constants with respect to $\omega\sqrt{\mu_0\varepsilon_0\varepsilon}$ rather than with respect to $\omega\sqrt{\mu_0\varepsilon_0}$):

$$2\mu_2^2 k_{y1} k_{y2} k_z^2 (1 - \cos k_{y1}b \cos k_{y2}b) + (N_1 k_x^2 + N_2) \sin k_{y1}b \sin k_{y2}b = 0\,, \tag{12}$$

where:

$$N_1 = -(\mu_1 - 1)^2 \frac{\mu_1^2 - \mu_2^2}{\mu_1} k_z^4 + \left[2\mu_1^2(\mu_1 - 1)^2 - 2\mu_1\mu_2^2(2\mu_1 - 1) + 2\frac{\mu_2^4}{\mu_1}(\mu_1 + 1)\right] k_z^2 +$$
$$- \mu_1^3(\mu_1 - 1)^2 + \mu_1\mu_2^2(\mu_1 - 1)(3\mu_1 - 1) - \mu_2^4(3\mu_1 - 2) + \frac{\mu_2^6}{\mu_1}\,,$$

$$N_2 = -(\mu_1 - 1)^2 k_z^6 + [3\mu_1(\mu_1 - 1)^2 - \mu_2^2(3\mu_1 - 1)] k_z^4 +$$
$$+ \left[-3\mu_1^2(\mu_1 - 1)^2 + \mu_2^2(3\mu_1 - 1)(2\mu_1 - 1) - \frac{\mu_2^4}{\mu_1}(3\mu_1 + 1)\right] k_z^2 +$$
$$+ \mu_1^3(\mu_1 - 1)^2 - \mu_1\mu_2^2(\mu_1 - 1)(3\mu_1 - 1) + \mu_2^4(3\mu_1 - 2) - \frac{\mu_2^6}{\mu_1}\,.$$

Equation (12) after some algebraic manipulations can be reduced to the one given originally by MIKAELIAN [5].

The field in the guide totally filled is given by (9), where the amplitudes A can be easily determined.

The solution still posesses reflection symmetry in the z direction and therefore the solution for the rectangular guide can be immediately found.

Equation (12) is now a relation between k_z^2 and k_x^2, since the odd powers of k_x appearing in equation (11) have disappeared. The structure is therefore reciprocal. However inspection of the solution (9) shows that there is not reflection symmetry along the x direction; but the solutions corresponding

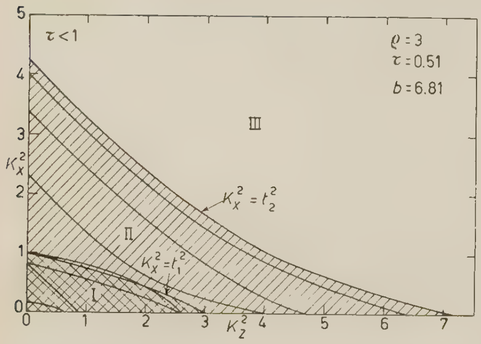

Fig. 2. – The mapping of the solutions of equation (12) for $\tau < 1$. The curves have been calculated for the values of ϱ, τ and b indicated. These values may be taken to correspond to

$$\varepsilon = 10; \qquad H = \frac{10^6}{4\pi}\ \text{A/m};$$

$$M_0 = 0.3\ \text{Wb/m}^2; \qquad f = \omega/2\pi = 1425\ \text{MHz};$$

$$b/\omega\sqrt{\mu_0\varepsilon_0\varepsilon} = 2.84''.$$

to $+k_x$ and $-k_x$ present specular symmetry with respect to the plane $y=b/2$. This lack of reflection symmetry accounts for the difficulty encountered in the solution of the problem relative to the rectangular guide with a d.c. magnetic field parallel to the axis of the guide.

Let us now discuss with some detail the modes distribution for the guide completely filled.

We shall suppose k_z^2 and k_x^2 to be real, and we shall discuss our solutions with reference to the positive quadrant of the k_z^2, k_x^2 plane. In other words we shall only consider those modes having unattenuated propagation along the z and x directions. This limitation follows from the fact that we are essentially interested in the problem of the rectangular guide transversely magnetized. Our modal solutions for such a guide will be obtained by letting $k_z = m\pi/a$, i.e. with real positive values of k_z^2. On the other hand only real positive values of k_x^2 are of interest for us, since real negative values correspond to modes under cut-off; and complex values would correspond to modes which individually do not carry power (7).

With reference to the y dependence we can classify the solutions (9)

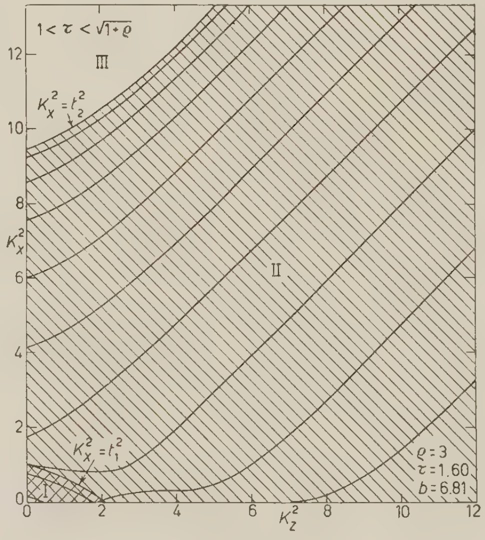

Fig. 3. – The same as Fig. 2 except for $1 < \tau < \sqrt{1+\varrho}$; $f = 4500$ MHz;

$$b/\omega\sqrt{\mu_0\varepsilon_0\varepsilon} = 0.9''.$$

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(7) G. BARZILAI: *Report* R/578/57, Polytechnic Institute of Brooklyn 506 (May 7, 1957), Brooklyn 1, New York.

into three classes: I) k_{y1} and k_{y2} both real; II) k_{y1} and k_{y2} one real and the other immaginary; III) k_{y1} and k_{y2} both immaginary. These three classes of solutions correspond to three zones of the k_z^2, k_x^2 plane. In virtue of equation (5) these three zones are limited by the two curves $k_x^2 = t_1^2$ and $k_x^2 = t_2^2$ which belong to an hyperbola, as it is easily recognized from equation (3). More exactly, zone II lies between the two curves, zone I and zone III lie respectively below and above zone II.

We notice now that all the points belonging to the two curves $k_x^2 = t_{1,2}^2$ (i.e. $k_{y1,2} = 0$) are solutions of equation (12). It is easily recognized, however, that these solutions correspond to zero amplitude fields.

It is convenient to study the behavior of our solutions taking τ as variable and keeping ϱ fixed.

The curve $k_x^2 = t_1^2$ intercepts the k_x^2 axis at the point $k_x^2 = 1$; it has an asymptote having a slope -1 and intercepting the k_x^2 axis at the point $k_x^2 = 1 + \varrho$. We can conclude therefore that this curve does not change much as τ varies.

The curve $k_x^2 = t_2^2$ intercepts the k_x^2 axis at the point $k_x^2 = (\mu_1^2 - \mu_2^2)/\mu_1$ and has an asymptote which has a slope equal to $-1/\mu_1$ and intercepting the k_x^2 axis at the point $k_x^2 = 1 + \mu_2^2/\mu_1(\mu_1 - 1)$.

This curve therefore changes considerably as τ varies, and consequently the configuration of the three zones varies.

Following this variation we shall divide the field of variability of τ into four regions, which correspond to the four typical mode configurations for structures of this type:

a) $\tau < 1$ (see Fig. 2). The curve $k_x^2 = t_2^2$ intercepts the k_x^2 axis at $k_x^2 > 1$ and the corresponding asymptote has a negative slope;

b) $1 < \tau < \sqrt{1+\varrho}$ (see Fig. 3). The curve $k_x^2 = t_2^2$ intercepts the k_x^2 axis at $k_x^2 > 1$, but the corresponding asymptote has a positive slope;

c) $\sqrt{1+\varrho} < \tau < 1 + \varrho$ (see Fig. 4). The curve $k_x^2 = t_2^2$ intercepts the k_x^2 axis at $k_x^2 < 0$, and the corresponding asymptote has a negative slope, and therefore this curve does not appear in the figure;

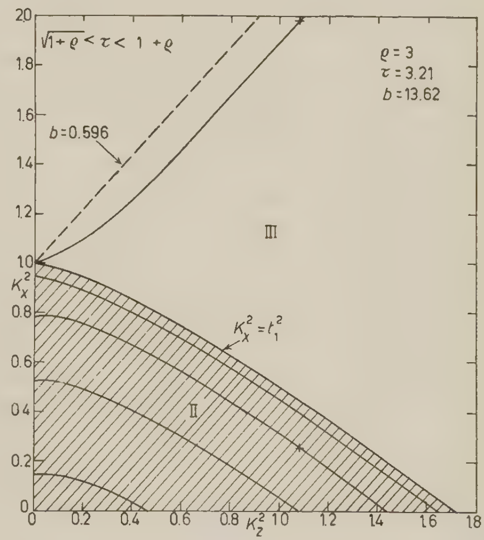

Fig. 4. – The same as Fig. 2 except for $\sqrt{1+\varrho} < \tau < 1 + \varrho$; $f = 9000$ MHz;

$$b/\omega\sqrt{\mu_0\varepsilon_0\varepsilon} = 0.9''.$$

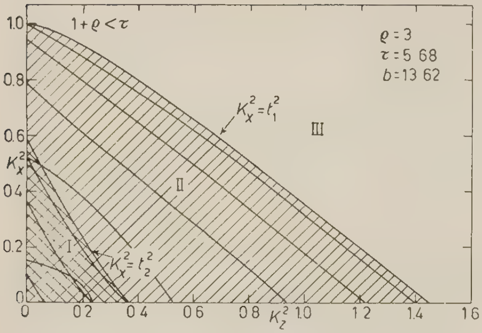

Fig. 5. – The same as Fig. 2 except for $\tau>1+\varrho$; $f=15\,900$ MHz; $b/\omega\sqrt{\mu_0\varepsilon_0\varepsilon}=0.51''$.

d) $\tau>1+\varrho$ (see Fig. 5). The curve $k_x^2=t_2^2$ intercepts the k_x^2 axis at $0<k_x^2<1$, and the corresponding asymptote has a negative slope.

Having now delimited the various zones, let us look at the distribution of the solutions inside these zones.

For $k_z=0$ from (12) we obtain:

$$k_x^2=t_{1,2}^2-\frac{n^2\pi^2}{b^2}\qquad n=0,1,2,\ldots \tag{13}$$

Equation (13) allows us to put down the starting points of the solution curves, which have the configuration depicted in Fig. 2, 3, 4, 5.

These curves have been calculated by assuming the values of ϱ, τ and b indicated in the figures.

If we keep ϱ and τ fixed and decrease b, the distance between the curves is increased, and the number of modal curves corresponding to propagating modes is reduced. Therefore if b is sufficiently small, we can say that: in the cases of Fig. 2 and 5 there will be only one solution curve which starts at $k_x^2=1$ and intercepts in some point the k_z^2 axis; in the case of Fig. 4 there will be only one solution curve which starts at $k_x^2=1$ and goes to infinity without intercepting the k_z^2 axis; in the case of Fig. 3 there will always exist an infinite number of solution curves.

The curves that go to infinity

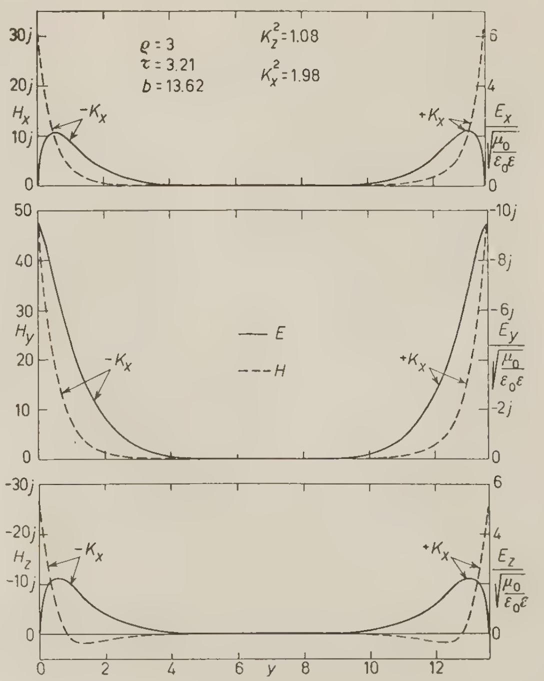

Fig. 6. – Field distribution for a mode of zone III. The calculations have been carried out for the values of k_z^2 and k_x^2 indicated by a star in Fig. 4. The lack of reflection symmetry in the x direction is shown in the figure by the drawing of the two configurations corresponding to $+k_x$ and $-k_x$.

in the zone II of the case of Fig. 3 have the same asymptotic direction as the curve $k_x^2 = t_2^2$. Mode solutions in the zone III exist only in the case of Fig. 4. The corresponding curve has an asymptote whose slope is $\mu_1/(\mu_2^2 - \mu_1^2)$.

In Fig. 6 and 7 the field configuration has been depicted for modes of the zones II and III. These modes correspond to the points indicated with a star in Fig. 4. In Fig. 6 the specularly symmetrical field corresponding to $-k_x$ has also been drawn. This drawing has been omitted in Fig. 7 for clarity.

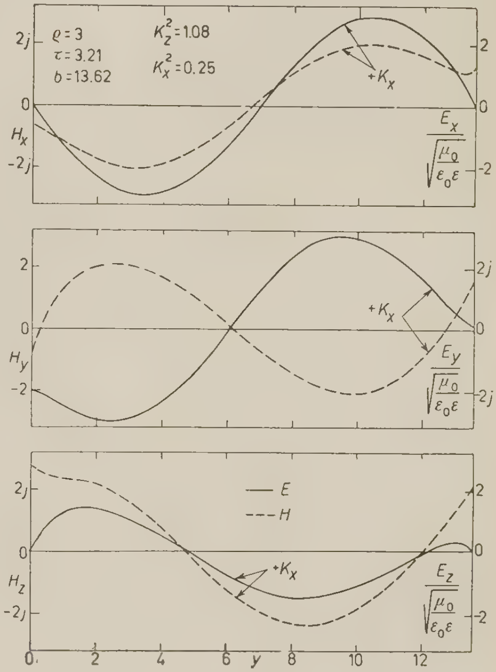

Fig. 7. – Field distribution for a mode of zone II. The calculations have been carried out for the values of k_z^2 and k_x^2 indicated by a star in Fig. 4. The specularly symmetrical configuration relative to $-k_x$ has been omitted for clarity.

From what has been said at the end of the last section, it is apparent that the diagrams of Fig. 2, 3, 4, 5 can be used to represent the mode distribution in a rectangular guide transversely magnetized. These modes are given by the interception of the various curves with vertical lines whose distances from the k_z^2 axis are $m^2\pi^2/a^2$, $m = 0, 1, 2, \ldots$.

For $m = 0$ it can be seen that the solutions corresponding to t_1^2 in (13) have zero amplitude.

In the cases of Fig. 2 and 5 there are a finite number of propagating modes, while in the cases of Fig. 3 and 4 this number is always infinite, irrespective of the values of a and b.

For rectangular guides of vanishingly small cross-section there are propagating modes only in the cases of Fig. 3 and 4. These modes have field distribution similar to those depicted respectively in Fig. 7 and 6. These modes are the same as those found by Seidel (6) using his asymptotic analysis.

The modes of Fig. 7 appear to utilize better the cross-section of the guide, but for a given ϱ they correspond to frequencies nearer the resonant frequency.

5. – Determination of the modes cut-off conditions for the completely filled guide.

We are now in a position to determine the modes cut-off conditions of a rectangular guide. Obviously this determination has a meaning only in the cases of Fig. 2 and 5 ($\tau < 1$ or $\tau > 1+\varrho$, i.e. $\mu_1^2 > \mu_2^2$), since in the cases of Fig. 3 and 4 for guides of any dimensions exist unattenuated propagating modes.

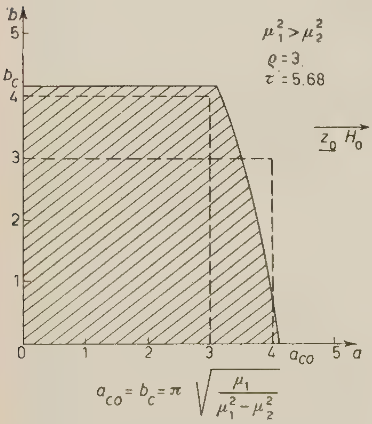

Fig. 8. – Diagram to determine the modes cut-off critical dimensions of a rectangular guide for the values of τ and ϱ corresponding to Fig. 5.

When b is sufficiently large, modes with $k_z = 0$ (TE modes) can propagate for any value of a. By decreasing b these modes go under cut-off when the critical value:

$$b_c = \pi \sqrt{\frac{\mu_1}{\mu_1^2 - \mu_2^2}}, \tag{14}$$

is reached.

For $b < b_c$ TE modes cannot propagate; however other modes can propagate for sufficiently large a, as it is apparent if one recalls what has been said in Sect. **4**. Therefore TE modes are not in general the lowest modes.

For each value of $b < b_c$ it is possible to determine a critical value of a, a_c, such that for $a < a_c$ no unattenuated propagating modes exist in the guide. This is equivalent to say that the vertical line, whose distance from the k_x^2 axis is $(\pi/a)^2$, should not cross any modal curve.

The value of a_c can be found from equation (12) by letting $k_x = 0$ and b equal to the chosen $b < b_c$.

In particular is found:

$$\lim_{b \to 0} a_c = a_{c0} = b_c\,.$$

Numerical calculations have been carried out for the case of Fig. 5 and the results are given in Fig. 8.

The diagram of Fig. 8 allows to decide by simple inspection if for a guide of a given cross-section will or will not exist unattenuated propagating modes. In fact, let two of the sides of the guide in question coincide with the a and b axes, if the vertex opposite to the origin 0 is internal to the shaded area all the modes are below cut-off; otherwise at least one mode is above cut-off.

For instance with reference to Fig. 8, it could be of interest to notice that a rectangular guide with $a = 3$ and $b = 4$ is below cut-off; while a guide with $a = 4$ and $b = 3$ is above cut-off and propagates only one mode, which incidentally is not a TE mode. In other words by rotating of 90° the magnetic field, for the same guide will or will not exist propagating modes.

6. – Conclusions.

For the case of a guide with a slab of ferrite against one side wall a general solution has been derived and the characteristic equation obtained.

For the special case of the rectangular guide completely filled some conclusions can be drawn from the preceding analysis.

a) In the cases of Fig. 3 and 4 an infinite number of modes propagate irrespectively to the dimensions of the guide.

b) In the cases of Fig. 2 and 5 there are some critical dimensions of the guide below which all our modes are under cut-off.

c) Modes corresponding to $k_z = 0$ (TE modes) are not in general the lowest modes as in the case of guides filled with isotropic medium. This is obvious for the cases of Fig. 3 and 4, but it may be also so for the other two cases.

d) The preceding analysis allows to calculate the modes cut-off dimensions for the rectangular guide.

RIASSUNTO (*)

Si ottiene una soluzione modale generale per una guida rettangolare con una lastra di ferrite appoggiata a una parete magnetizzata in direzione parallela a questa parete e normalmente all'asse della guida e se ne ottiene l'equazione caratteristica. Specializzando questa equazione per il caso della guida completamente riempita si deriva la relativa equazione caratteristica. Si discute dettagliatamente quest'ultima equazione, facendo l'analisi delle possibili configurazioni del campo. Si eseguono calcoli numerici per chiarire la distribuzione dei modi. Si impiega, finalmente, la precedente analisi per derivare le dimensioni del taglio dei modi per la guida completamente riempita.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

On the Spin of the Muon.

L. M. Brown (*)

Physics Department, Northwestern University - Evanston, Illinois

V. L. Telegdi

The Enrico Fermi Institute for Nuclear Studies, The University of Chicago - Chicago, Illinois

(ricevuto il 22 Novembre 1957)

Summary. — The distribution in angle and energy of the electron arising from the decay of a longitudinally polarized muon of spin $\frac{3}{2}$ is calculated, using the most general parity nonconserving direct interaction. The result depends upon three parameters (functions of the coupling constants) which cannot be adjusted to give agreement with experiment.

1. – Introduction.

Using the Dirac-Fierz-Pauli (DFP) equations (1) for particles of half-integral spin, it has been shown that such particles have (neglecting radiative corrections) a gyromagnetic ratio equal to the reciprocal of the spin (2). Recent measurements of the gyromagnetic ratio of the muon (3) have, according to this theory, the unique interpretation that the muon spin is $\frac{1}{2}$. This measurement shows also that the muon cannot be the spin $\frac{3}{2}$ state of the dibaric system proposed by Bhabha (4). Other electromagnetic evidence is in accord

(*) Work supported in part by the National Science Foundation (USA).

(1) P. A. M. Dirac: *Proc. Roy. Soc.* (*London*), A **155**, 447 (1936); M. Fierz: *Helv. Phys. Acta*, **12**, 3 (1938); M. Fierz and W. Pauli: *Proc. Roy. Soc.* (*London*), A **173**, 211 (1939).

(2) P. A. Moldauer and K. M. Case: *Phys. Rev.*, **102**, 279 (1956). This result has been proven previously for spin $\frac{3}{2}$ by F. J. Belinfante: *Phys. Rev.*, **92**, 997 (1953).

(3) T. Coffin, R. L. Garwin, L. M. Lederman, S. Penman and A. M. Sachs: *Phys. Rev.*, **106**, 1108 (1957).

(4) H. J. Bhabha: *Phil. Mag.*, **43**, 33 (1952); R. L. Pease and J. Pease: *Phys. Rev.*, **107**, 1169 (1957).

with the muon being a normal Dirac spin $\frac{1}{2}$ particle, although the disproof of the possibility of spin $\frac{3}{2}$ is made difficult by the non-uniqueness of the electromagnetic interaction, even if consideration is restricted to the DFP particles (5).

Nevertheless, we believe that it is of some interest to investigate the spin of the muon by a method in which no assumptions are made concerning the form of the electromagnetic interaction. Experiments on the decay of *longitudinally polarized* muons can be used to determine the spin using for theoretical comparison only the « free field » properties of the particles and the theory of β-interaction. Previous attempts to distinguish between the possibilities $S=\frac{1}{2},\frac{3}{2}$ allowed the reaction $\pi\to\mu+\nu$, based on the shape of the spectrum of decay electrons from *unpolarized* muons, have been inconclusive since this spectrum can be fitted equally well by assuming either spin (6-8).

We will show that the most general parity non-conserving decay of a spin $\frac{3}{2}$ muon (not involving derivative couplings) cannot be reconciled with the observed angular distribution of the decay electrons from muons at rest when the muon has the degree of polarization required by the observed asymmetry (9). The theoretical assumptions of the present investigation imply little more than mere Lorentz invariance.

The results we shall obtain will be modified by radiative corrections. If these corrections were as small as in the spin $\frac{1}{2}$ case (10,11) they should not seriously affect our conclusions. If they were, on the other hand, much larger for the hypothetical spin $\frac{3}{2}$ particle, then the early arguments ruling out $S=\frac{3}{2}$ on the basis of electromagnetic evidence would already have been valid.

2. – Calculation.

The most general Fermi interaction for the spin $\frac{3}{2}$ muon decay, not involving gradient couplings, can be written (12) (for distinguishable neutrinos):

$$H'=\sum_{V,T,A} m_j\left[g_j(\bar{\nu}O_j\nu)+g_j'(\bar{\nu}O_j(i\gamma_5)\nu)\right]+\text{herm. conj.} \tag{1}$$

(5) The magnetic moment, however, is free of arbitrary parameterization (see ref. (2))

(6) R. E. Behrends and C. Fronsdal: *Phys. Rev.*, **106**, 345 (1957).

(7) E. R. Caianiello: *Phys. Rev.*, **83**, 735 (1951).

(8) L. Rosenson: *Doctoral Dissertation, University of Chicago* (1957).

(9) R. L. Garwin, L. M. Lederman and M. Weinrich: *Phys. Rev.*, **105**, 1415 (1957); J. I. Friedman and V. L. Telegdi: *Phys. Rev.*, **105**, 1681 (1957).

(10) T. Kinosita and A. Sirlin: *Phys. Rev.*, **107**, 593 (1957).

(11) R. E. Behrends, R. J. Finkelstein and A. Sirlin: *Phys. Rev.*, **101**, 866 (1956).

(12) T. D. Lee and C. N. Yang: *Phys. Rev.*, **104**, 254 (1956). We use the relativistic notation of R. P. Feynman: *Phys. Rev.*, **76**, 769 (1949).

with

$$O_j = \gamma_\varrho\,, \quad -(i/2\sqrt{2})(\gamma_\varrho\gamma_\sigma - \gamma_\sigma\gamma_\varrho)\,, \quad -i\gamma_\varrho\gamma_5 \tag{1a}$$

and

$$\left\{\begin{aligned} m_V &= (\bar{e}\mu_\varrho)\,,\\ m_T &= -(i/2\sqrt{2})\left[(\bar{e}\gamma_\varrho\mu_\sigma) - (\bar{e}\gamma_\sigma\mu_\lambda)\right],\\ m_A &= i(\bar{e}\gamma_5\mu_\varrho)\,. \end{aligned}\right. \tag{1b}$$

The electron and neutrino field operators are, respectively, e and ν. The entity μ_ϱ refers to the muon and its subscript is a vector index. We adopt the Rarita-Schwinger representation of the spin $\frac{3}{2}$ theory [13] and write the interaction in « charge retention » order for convenience. All other orderings are physically equivalent, involving merely a redefinition of coupling constants [6,14]. The mass of the electron will be neglected.

A convenient set of orthonormal positive energy spin states have been determined by KUSAKA [15]. In the rest frame these are [16]:

$$\left\{\begin{aligned} \mu_\varrho^{(1)} &= h_\varrho^{(1)}u^+\,, \quad \mu_\varrho^{(2)} = h_\varrho^{(2)}u^-\,,\\ \mu_\varrho^{(3)} &= (h_\varrho^{(2)}u^+ - \sqrt{2}\,h_\varrho^{(3)}u^-)/\sqrt{3}\,,\\ \mu_\varrho^{(4)} &= (h_\varrho^{(1)}u^- + \sqrt{2}\,h_\varrho^{(3)}u^+)/\sqrt{3}. \end{aligned}\right. \tag{2}$$

Here $u^\pm$ are positive energy Dirac free particle spinors for spin parallel and antiparallel to the quantization axis (z-axis) and the $h_\varrho^{(i)}$ are unit four-vectors:

$$\left\{\begin{aligned} h_\varrho^{(1)} &= \{1,\ i,\ 0,\ 0\}/\sqrt{2}\,,\\ h_\varrho^{(2)} &= \{1,\ -i,\ 0,\ 0\}/\sqrt{2}\,,\\ h_\varrho^{(3)} &= \{0,\ 0,\ 1,\ 0\}\,. \end{aligned}\right. \tag{2a}$$

Using (2) the μ-e matrix elements for an arbitrary muon spin state are expressed in terms of spin $\frac{1}{2}$ matrix elements of lower tensor order (see Appendix). The neutrino matrix elements are obviously identical with the spin $\frac{1}{2}$

([13]) W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1940).

([14]) L. MICHEL: *Proc. Phys. Soc.* (*London*), A **63**, 514 (1949).

([15]) S. KUSAKA: *Phys. Rev.*, **60**, 61 (1940). They are quoted in ref. ([7]).

([16]) Clearly these are nothing but the linear combinations of products of spin 1 and spin $\frac{1}{2}$ functions which belong to the total spin $\frac{3}{2}$, but the relativistic notation is retained for convenience.

case. Indeed, as a useful check on these calculations the general parity non-conserving spin $\frac{1}{2}$ result of previous authors ([17,10]) has been obtained.

Using the usual trace techniques, we find that we can write $|\langle H'\rangle|^2$ in the form

$$|\langle H'\rangle|^2 = aP_{VV}N_{VV} + bP_{TT}N_{TT} + a\xi P_{VA}N'_{VA} + b\eta P_{TT}N'_{TT}\,, \tag{3}$$

where

$$\left\{\begin{aligned} a &= |g_V|^2 + |g'_V|^2 + |g_A|^2 + |g'_A|^2 \\ b &= |g_T|^2 + |g'_T|^2 \\ a\xi &= 2\,\mathrm{Re}\,(g_V\bar{g}'_A + g_A\bar{g}'_V) \\ b\eta &= 2\,\mathrm{Re}\,g_T\bar{g}'_T \end{aligned}\right. \tag{3a}$$

and

$$P_{VV} = \bar{m}_V m_V\,, \quad \text{etc.} \tag{3b}$$

The factors N_{ij}, N'_{ij} are the tensors obtained on squaring the neutrino part of the matrix element. Terms which vanish upon integration to obtain the electron spectrum are omitted.

We shall discuss first the case of a muon completely polarized in the positive z-direction. Writing p, k, k' for the momenta, respectively, of the electron and the two neutrinos and

$$\left\{\begin{aligned} (VV) &= (2pkk')P_{VV}N_{VV}\,, \\ (VA)' &= (2pkk')P_{VA}N'_{VA}\,, \quad \text{etc.} \end{aligned}\right. \tag{4}$$

we obtain

$$\left\{\begin{aligned} (VV) &= (AA) = P(kk' - k_z k'_z) \\ 2(TT) &= k(pk' - p_z k'_z) + k'(pk - p_z k_z) - p(kk' - k_z k'_z) \\ (VA)' &= (AV)' = -(p_z/p)(VV) \\ 2(TT)' &= -p(kk'_z + k'k_z) + p_z(kk' + k_z k'_z)\,. \end{aligned}\right. \tag{5}$$

The differential electron spectrum is

$$P(x, \vartheta)\,\mathrm{d}x\,\mathrm{d}\Omega = Nx^2\,\mathrm{d}x\,\mathrm{d}\Omega\,R(x, \vartheta)\,, \tag{6}$$

([17]) C. Bouchiat and L. Michel: *Phys. Rev.*, **106**, 170 (1957); S. Larsen, E. Lubkin and M. Tausner: *Phys. Rev.*, **107**, 856 (1957).

where x is the ratio of the electron momentum to its maximum value and N is a normalization constant. From (5) we get

$$(7)\qquad R(x,\vartheta) = 2\left(1 - x + \frac{x^2}{8}\sin^2\vartheta\right)(1 - \xi\cos\vartheta) + \\ + \frac{b}{2a}\left[1 - \left(x - \frac{x^2}{4}\right)\sin^2\vartheta + \eta\cos\vartheta\left(1 - \frac{x^2}{4}\sin^2\vartheta\right)\right],$$

with $\cos\vartheta = p_z/p$ and a, b, ξ, η given by Eq. (3a). The result (7) holds for any state of complete polarization with appropriate rotation of the quantization axis.

The electron energy spectrum is obtained by dropping odd terms in $\cos\vartheta$ and replacing $\sin^2\vartheta$ by its average value $\frac{2}{3}$. We obtain

$$(8)\qquad \bar{R}(x) = 2\left(1 - x + \frac{x^2}{12}\right) + \left(\frac{b}{2a}\right)\left(1 - \frac{2}{3}x + \frac{x^2}{6}\right),$$

which can be put (aside from a constant) in the form of the one-parameter formula obtained by BEHRENDS and FRONSDAL (6) for the general parity conserving interaction, namely

$$(9)\qquad \bar{R}_{BF} = \left(1 - x + \frac{x^2}{12}\right) + \varrho'\left(\frac{4}{3}x - 1\right),$$

by the correspondence

$$(10)\qquad \varrho' = \tfrac{1}{2}b/(2a + b)\,.$$

The parameter ϱ' is the spin $\frac{3}{2}$ analogue of Michel's (14) spin $\frac{1}{2}$ parameter ϱ. Experiment gives equally good agreement (6,8) with $\varrho = 0.68 \pm 0.05$ (including radiative corrections), or with $\varrho' = 0.26 \pm 0.04$.

From Eq. (10) we see that the absence of tensor interaction ($b = 0$) correponds to $\varrho' = 0$. This is the prediction of the two-component neutrino theory (18) for the decay $\mu \to e + \nu + \bar{\nu}$, so that for that theory the choice of spin $\frac{3}{2}$ for the muon is unacceptable. On the other hand, for identical neutrinos antisymmetrization leads to conditions equivalent to

$$(11)\qquad g_V = g_T = g'_T = g'_A = 0\,.$$

Thus in the two-component theory, the decay of a spin $\frac{3}{2}$ muon into identical

(18) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957); A. SALAM: *Nuovo Cimento*, **5**, 299 (1957); L. LANDAU: *Nuclear Phys.*, **3**, 127 (1957).

neutrinos cannot occur at all, since it can proceed only via the tensor interaction and this vanishes.

To obtain a ϱ' in agreement with experiment, we must include at least T and either V or A, taking $b \approx 2a$. The decay spectrum from a completely polarized muon then becomes, from Eq. (7):

$$R(x, \vartheta) = 3 - 2x - \left(x - \frac{x^2}{2}\right)\sin^2\vartheta + \cos\vartheta\left[\eta\left(1 - \frac{x^2}{4}\sin^2\vartheta\right) - 2\xi\left(1 - x + \frac{x^2}{8}\sin^2\vartheta\right)\right]. \tag{12}$$

As the experimental electron distribution from polarized muons at rest, averaged over energies, is well represented by an angular distribution of the form [9,19]

$$1 + A\cos\vartheta\,, \tag{13}$$

with $A = -0.26 \pm 0.02$, we shall take $\xi = -\eta$ to eliminate the term in $\sin^2\vartheta$ from the odd part of the distribution. Thus,

$$R(x, \vartheta) = (3 - 2x)(1 + \eta\cos\vartheta) - \left(x - \frac{x^2}{2}\right)\sin^2\vartheta\,. \tag{14}$$

Integrating Eq. (6) over the electron momentum x, we obtain (inserting Eq. (14)) the angular distribution

$$f(\vartheta) \sim (1 + \eta\cos\vartheta) - 0.3\sin^2\vartheta\,. \tag{15}$$

While it is possible to make η assume the value -0.26, the term $0.3\sin^2\vartheta$ definitely rules out this theory since the experiments do not permit its presence. Even if one should choose to ignore this term, the energy dependence of the asymmetry as given in Eq. (14) would contradict experiment grossly [19] and rule out this theory on another ground.

The term in $\sin^2\vartheta$ must be present, with the opposite sign, in the polarization state for which $S_z = \frac{1}{2}$. This can be inferred from the fact that the sum over all four polarization states must yield the unpolarized result. But reversing the direction of the z-axis has the effect only of changing the signs of the odd powers of $\cos\vartheta$.

([19]) D. Berley, T. Coffin, R. L. Garwin, L. M. Lederman and M. Weinrich: *Phys. Rev.*, **106**, 835 (1957); S. C. Wright: *Proceedings of the 1957 Rochester Conference*; see also D. H. Wilkinson: *Nuovo Cimento*, **6**, 516 (1957) for a survey of relevant data.

The case of an arbitrary state of muon polarization is considered in the Appendix. However, it should be pointed out that the case treated above of muons fully polarized along their momenta is the only one of physical interest when they arise from pion decay, since cases of complete polarization in some other direction cannot be produced for reasons of symmetry. The case of partial longitudinal polarization, and incoherent superposition of $S_z = +\frac{3}{2}$ and $S_z = -\frac{3}{2}$ can lead only to a reduction of the asymmetry without affecting the coefficient of the $\sin^2\vartheta$ term.

Appendix

We shall state briefly here some results for an arbitrary state of muon polarization. Such a state is described by a linear combination of the $\mu_\varrho^{(i)}$ given in (2) and hence by

$$\mu_\varrho = h_\varrho u^+ + f_\varrho u^- , \tag{16}$$

with

$$\bar{h}_\varrho h_\varrho + \bar{f}_\varrho f_\varrho = -1 \quad \text{and} \quad h_4 = f_4 = 0 . \tag{16a}$$

(The consequent normalization of μ_ϱ to -1 is obviously of trivial significance.)
Substitution of (16) into (1*b*) gives

$$\left\{\begin{aligned} m_V &= h_\varrho\langle S\rangle_+ + f_\varrho\langle S\rangle_- ,\\ m_A &= ih_\varrho\langle P\rangle_+ + if_\varrho\langle P\rangle_- ,\\ m_T &= (i/2\sqrt{2})[\langle V_\lambda\rangle_+ h_\varrho - \langle V_\varrho\rangle_+ h_\lambda + \langle V_\lambda\rangle_- f_\varrho - \langle V_\varrho\rangle_- f_\lambda] ,\end{aligned}\right. \tag{17}$$

where

$$\langle S\rangle_\pm = (\bar{e}O_S u^\pm) , \quad \text{etc.} ,$$

and

$$O_S = 1 , \qquad O_P = \gamma_5 , \qquad O_V = \gamma_\varrho .$$

With this reduction, the usual trace and projection operator methods for spin $\frac{1}{2}$ particles can be employed. The P_{ij} of Eq. (3) are found to be in the case of general polarization:

$$2pP_{VV} = 2pP_{AA} = p(\bar{h}_\varrho h_\lambda + \bar{f}_\varrho f_\lambda) , \tag{18a}$$

$$\begin{aligned} 4pP_{TT} = {} & (\bar{h}_\mu h_\sigma + \bar{f}_\mu f_\sigma)U_{\lambda\varrho} + i(\bar{h}_\mu h_\sigma - \bar{f}_\mu f_\sigma)p_\tau\varepsilon_{\lambda\tau\varrho 3} - \\ & - (\bar{h}_\mu f_\sigma - \bar{f}_\mu h_\sigma)p_\tau\varepsilon_{\lambda\tau\varrho 2} + i(\bar{h}_\mu f_\sigma + f_\mu h_\sigma)\cdot \\ & \cdot\{p(g_{\lambda 2}g_{\varrho 3} - g_{\lambda 3}g_{\varrho 2}) + p_3(g_{\lambda 2}g_{\varrho 4} - g_{\varrho 2}g_{\lambda 4}) - p_2(g_{\lambda 3}g_{\varrho 4} - g_{\varrho 3}g_{\lambda 4})\} ,\end{aligned} \tag{18b}$$

with

$$U_{\lambda\varrho} = p_\lambda g_{\varrho 4} + p_\varrho g_{\lambda 4} - p g_{\lambda\varrho} .$$

$$(18c) \qquad 2pP_{VA} = 2pP_{AV} = P_3(\bar{h}_\varrho h_\lambda - \bar{f}_\varrho f_\lambda) + \bar{h}_\varrho f_\lambda(p_1 - ip_2) + \bar{f}_\varrho h_\lambda(p_1 + ip_2) .$$

The expression (18b) should be antisymmetrized in (λ, μ) and (ϱ, σ) but this is unnecessary if N_{TT} and N'_{TT} in Eq. (3) are so antisymmetrized. The mass of the electron is neglected, its four-momentum being $p_\mu = (\boldsymbol{p}, p)$. The spin quantization is referred to the 3-axis. The electron momentum $\boldsymbol{p}$ is taken to lie in the (1, 3) plane.

RIASSUNTO (*)

Si calcola la distribuzione angolare ed energetica dell'elettrone derivante dal decadimento di un muone polarizzato longitudinalmente con spin $\frac{3}{2}$, servendosi della più generale interazione diretta non conservante la parità. Il risultato dipende da tre parametri (funzioni delle costanti d'accoppiamento) che non sono suscettibili di aggiustamento per accordare il risultato con l'esperienza.

(*) *Traduzione a cura della Redazione.*

Determination of the Coupling Constants in $\mu^- + p \to n + \nu$.

L. WOLFENSTEIN (*)
CERN - Genève

(ricevuto il 2 Dicembre 1957)

Summary. — For the reaction $\mu^- + p \to n + \nu$ the results of possible experimental observations including the neutron polarization have been calculated on the basis of the two-component neutrino theory. In the non-relativistic approximation these observations overdetermine the coupling constants, but relativistic effects, which are included in the calculation, are quite important for certain values of the coupling constants.

The recent developments in the theory of weak interactions increase the interest in determining the detailed character of the coupling responsible for the reaction $\mu^- + p \to n + \nu$. We have calculated the results of possible experimental observations on this reaction including the neutron polarization assuming that 1) it is entirely due to a local interaction between the four fields as in the Fermi theory of β-decay, 2) the neutrino is described by the two-component theory, 3) the initial system of μ-meson and proton can be treated non-relativistically, and 4) effects of virtual π-mesons can be neglected. Relativistic effects, depending on v/c for the final neutron have been included and are quite important for certain values of the coupling constants.

The results are:

1) The rate for the capture process is (with $\hbar = c = 1$)

$$\tau^{-1} = \frac{p_\nu^2}{2\pi^2 a_\mu^3} \frac{I_0}{1 + \lambda^2}, \tag{1}$$

$$I_0 = |G_F|^2 + 3|G_G|^2 + 2\lambda \operatorname{Re}\left[G_G^*(\Delta_A - 2\Delta_V) + G_F^*\Delta_V\right] + \lambda^2(|\Delta_A|^2 + 3|\Delta_V|^2), \tag{1a}$$

(*) On leave of absence from Carnegie Institute of Technology, Pittsburgh, Pa., USA.

2) The angular distribution or asymmetry of the neutrons [1,2] is given by

$$I = I_0(1 + PA\boldsymbol{p}\cdot\boldsymbol{s})\,, \tag{2}$$

$$I_0 A = |G_F|^2 - |G_G|^2 + 2\lambda\,\mathrm{Re}\,G_G^*(\Delta_A + 2\Delta_V) + G_F^*\Delta_V] + \lambda^2(|\Delta_A|^2 - |\Delta_V|^2) \tag{2a}$$

3) The spin polarization $\langle\boldsymbol{\sigma}_n\rangle$ of the neutrons is given by

$$I\langle\boldsymbol{\sigma}_n\rangle = \boldsymbol{p}(a + Pb\boldsymbol{p}\cdot\boldsymbol{s}) + Pc\boldsymbol{s} + Pd(\boldsymbol{p}\times\boldsymbol{s})\,, \tag{3}$$

$$a = 2\{\mathrm{Re}\,(G_G^* G_F) - |G_G|^2 + \lambda\,\mathrm{Re}(G_F^*\Delta_A + 3G_G^*\Delta_V) + \\ + \lambda^2[\mathrm{Re}\,(\Delta_A\Delta_V^*) - |\Delta_V|^2]\}\,, \tag{3a}$$

$$b = 2\lambda\,\mathrm{Re}\,[(G_F - G_G)^*(\Delta_A + \Delta_V)] + 4\lambda^2[\mathrm{Re}\,(\Delta_A\Delta_V^*) + |\Delta_V|^2]\,, \tag{3b}$$

$$c = 2\{\mathrm{Re}\,(G_G^* G_F) + |G_G|^2 + \lambda\,\mathrm{Re}\,(G_G^*\Delta_A - G_F^*\Delta_V) - \\ - \lambda^2[\mathrm{Re}\,(\Delta_A\Delta_V^*) + |\Delta_V|^2]\}\,, \tag{3c}$$

$$d = 2\,\mathrm{Im}\{G_G^* G_F + \lambda[G_F^*\Delta_V + G_G^*(\Delta_A + 2\Delta_V)] - \lambda^2\Delta_A\Delta_V^*\}\,. \tag{3d}$$

The symbols are

$G_F = C_S + C_V$,
$G_G = C_T + C_A$,
$\Delta_V = C_V - C_T$,
$\Delta_A = C_A - C_P$,
$\lambda = (v/c)(\gamma/\gamma+1) \doteq \frac{1}{2}(v/c) = .053$,
v = neutron velocity $(\gamma = [1 - v^2/c^2]^{-\frac{1}{2}})$,
P = μ-meson polarization,
$\boldsymbol{s}$ = unit vector in direction of μ-meson polarization,
$\boldsymbol{p}$ = unit vector in direction of neutrino momentum,
= — (unit vector in direction of neutron momentum),
p_ν = neutrino momentum,
a_μ = Bohr radius of μ-meson in hydrogen.

(1) T. D. Lee: in *Proc. of Rochester Conference* (1957); K. Huang, C. N. Yang and T. D. Lee: to be published.

(2) H. Überall: *Nuovo Cimento*, **6**, 533 (1957). The non-relativistic expression for A is given here for the four-component theory.

and C_i are the usual coupling constants [3]. We have assumed here that the emitted neutrino is right-handed. For a left-handed neutrino the results are the same except that the signs of A, a, and d must be reversed.

We look first at the non-relativistic results ($\lambda = 0$), which should ordinarily be accurate within about 10%. There are then only two coupling constants: the Fermi coupling G_F and the Gamow-Teller coupling G_G. We consider experiments designed to determine the absolute magnitude and relative phases of G_F and G_G as well as to test the validity of the assumptions. 1) The rate (Eq. (1)) together with the asymmetry (Eq. (2)) determine $|G_F|$ and $|G_G|$. 2) The neutron polarization parallel to its direction of motion provides a sensitive measure of the relative phase α of G_F and G_G; from Eqs. (3) with $\lambda = 0$ we have

$$-\langle \boldsymbol{\sigma}_n \rangle \cdot \boldsymbol{p} = 2\,\frac{1 - r' - P\cos\theta(1 + r')}{3 + r^2 + P\cos\theta(r^2 - 1)}\,, \tag{4}$$

where $\cos\theta = \boldsymbol{p}\cdot\boldsymbol{s}$, $r = |G_F|/|G_G|$, and $r' = r\cos\alpha$.

Although the measurement of the longitudinal polarization is difficult since a magnetic field is required to turn the spin before detection by scattering, it has the advantage that large values are possible even if the μ-mesons are completely depolarized. If the depolarization in hydrogen is very large this might be the best way to directly observe parity non-conservation in this process. Furthermore, in the special case that $G_F = -G_G$, the asymmetry vanishes while the longitudinal polarization is 100%. In Fig. 1 the longitudinal polarization is shown as a function of r and α for the case of unpolarized μ-mesons, which is the same as the average over all angles θ for the case of polarized μ-mesons. If r has been determined from the asymmetry or some other method (e.g., the capture rate in deuterium [4]) one sees that even a fairly rough measurement of the longitudinal polarization can distinguish between different values of α. Even if r is not known, if one assumes invariance under time reversal (that is, $\cos\alpha = \pm 1$), this measurement might put useful limits on the value of r and might determine the relative sign of G_G and G_F. 3) The transverse components of the neutron polarization are more directly detected but have the disadvantage of being proportional to the μ-meson polarization P, since the vector $\boldsymbol{s}$ is needed to define a transverse direction. In principle, the measurements discussed previously completely determine G_F and G_G and thus the transverse components of polarization, so that this additional measurement

(3) The interaction Hamiltonian is obtained from the hermitian conjugate (divided by $\sqrt{2}$) of Eq. (10*a*) of T. D. Lee and C. N. Yang: *Phys. Rev.*, **105**, 1671 (1957) with e replaced by μ.

(4) A. Rudik: *Dokl. Akad. Nauk SSSR*, **92**, 139 (1953); H. Primakoff: private communication.

could provide a check of the assumptions. One may note also that a non-zero component of polarization normal to the plane defined by $\boldsymbol{p}$ and $\boldsymbol{s}$ (that is, $d \neq 0$) would be direct evidence that the interaction is not invariant under

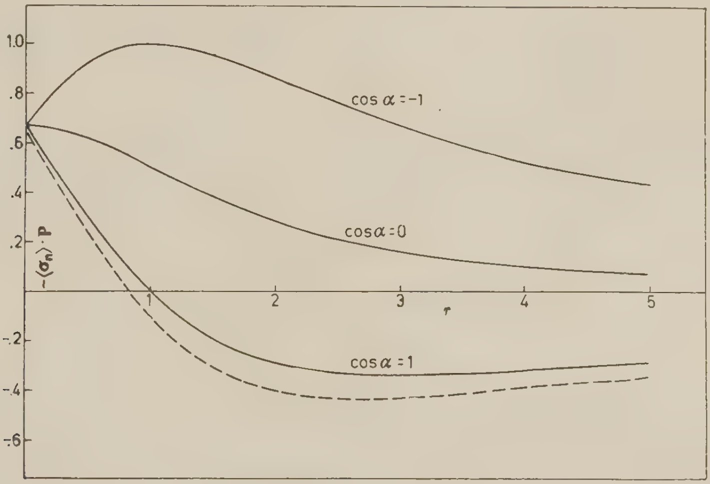

Fig. 1. – Polarization of neutrons parallel to their direction of motion as a function of r, the ratio of $|G_F|$ to $|G_G|$, for fixed values of the relative phase α for unpolarized μ-mesons. The solid curves are the non-relativistic approximation. The dashed curve gives the relativistic result for $\cos\alpha = +1$ for the special case of only vector and axial-vector interactions.

time-reversal. Of course, the determination of the phase α discussed above is in itself a check of time-reversal invariance, but it is indirect in that it depends on the assumptions and the non-relativistic approximation.

The relativistic effects must be included for quantitative conclusions and in the special cases that C_T is close to $-C_A$ or C_S is close to $-C_V$ for qualitative conclusions as well. We illustrate these effects by considering the asymmetry A for special cases: 1) Pure Fermi interaction. In this case A is 1.0 as long as terms of order λ^2 are neglected. 2) Pure Gamow-Teller interaction. For a pure tensor interaction, A equals $-.38$ and for a pure axial vector $-.29$, whereas non-relativistically A equals $-.33$. In Fig. 2, A is plotted as a function of $C_A/(C_T + C_A)$ assuming this is real. The curve is designed to illustrate the extreme deviations from the non-relativistic value that are possible; however, these correspond to the unlikely situation that the tensor and axial-vector interactions have a large destructive interference so that $|C_A + C_T|$ is much less than $|C_A|$. 3) $|G_F| = |G_T|$. Assuming two of the coupling constants are zero (as well as the pseudoscalar) the asymmetry is given in Table I and is to be compared with the non-relativistic value of zero. For the case $C_A = C_V$ we have also shown the relativistic effects on the longitudinal

polarization in Fig. 1 (for $C_A = -C_v$ the effects turn out to be much smaller).

It is worthwhile to make a few general comments on the neutrons from the capture of μ-mesons in complex nuclei, since these are more readily observed.

TABLE I. – *Asymmetry parameter A for $|G_F| = |G_T|$ for different coupling combinations.*

non-zero couplings	+	—
$C_A = \pm C_v$	.105	0
$C_A = \pm C_s$	.026	.026
$C_T = \pm C_v$	0	—.048
$C_T = \pm C_s$	—.077	—.024

In this case the relativistic effects may be two or three times larger in magnitude both because it is desirable to look at the faster neutrons and because the target protons are also in motion. Of course, there may be more important

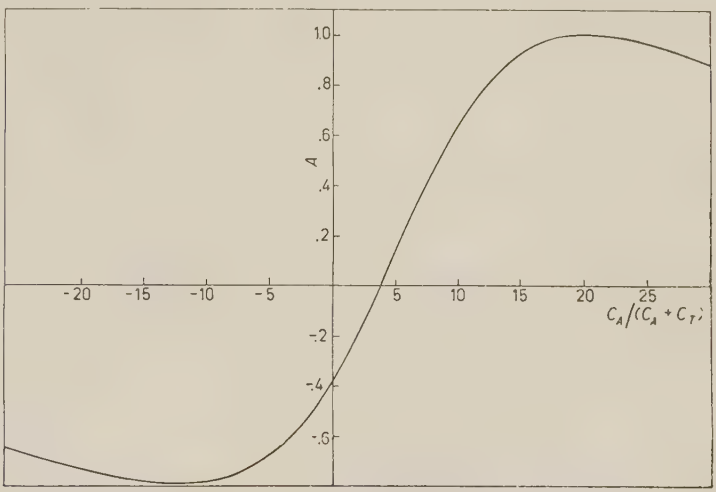

Fig. 2. – Asymmetry parameter A including relativistic effects as a function of $C_A/(C_T+C_A)$ assuming a pure Gamow-Teller interaction.

effects of the target proton motion (2) as well as possibly large effects of the nuclear interactions of the outgoing neutrons. These effects will reduce the asymmetry and the longitudinal polarization of the neutrons, but a measurement of either of these may still be used to give direct evidence of parity non-

conservation in μ-capture. On the other hand, the final-state interaction may induce transverse components of polarization so that no direct evidence on time-reversal invariance seems possible except for capture in hydrogen.

Note added in proof.

After completing this work, I have noted an article on the same subject by I. S. Shapiro, E. I. Dolinsky and L. D. Blohincev in *Nuclear Physics*, **4**, 273 (1957). This article includes our Eq. (2*a*) for the asymmetry and gives the longitudinal polarization of the neutron for pure interactions.

RIASSUNTO (*)

In base alla teoria del neutrino a due componenti, si sono calcolati per la reazione $\mu^-+p\to n+\nu$ i risultati delle possibili osservazioni sperimentali, compresa la polarizzazione dei neutroni. In approssimazione non relativistica tali osservazioni sovradeterminano le costanti di accoppiamento, ma gli effetti relativistici inclusi nel calcolo hanno importante influenza su alcuni valori delle costanti di accoppiamento.

(*) *Traduzione a cura della Redazione.*

Note on the Track Density of Minimum-Ionizing Particles in Large Emulsion Stacks.

B. Bhowmik (*), J. H. Davies, D. Evans and D. J. Prowse
H. H. Wills Physical Laboratory, University of Bristol

(ricevuto l'8 Gennaio 1958)

Summary. — On the basis of recent experience, it is suggested that Ilford G-5 emulsion stacks should be refrigerated after exposure when they are to be shipped over large distances by public transport.

A large quantity of Ilford G5 emulsion exposed to artificially accelerated particles has been processed at Bristol in recent years and our experiences may be of interest to other laboratories processing similar large emulsion stacks. It has long been known that high temperatures and humidities can induce a fading of the latent image, but until recently we were not convinced that such factors were responsible for the apparent low grain densities obtained in numerous stacks.

For a variety of reasons it is generally desirable to have a high value of minimum ionization in large stacks of stripped emulsions exposed to the beams from the high energy machines.

To obtain such a value recent development procedure has been a hot plate for one hour at some 27 °C with no KBr restrainer. The minimum ionization values obtained in stacks exposed to the beams from the Berkeley Bevatron since 1955 are shown in Table I. The stacks were all greater than 2 litres in volume and the figures given are mean values obtained from many plates. Because of slight variation from plate to plate it is given to the nearest .5 blob. The variance does not exceed this value. A common processing technique was employed for all stacks.

The stacks were sent to Bristol by normal air-transport with a stop-over in New York for trans-shipment. The total transit time was about 14 days.

No significant difference has been observed between the first and last processed portions of the same stack. Stacks awaiting processing at the laboratory are always carefully stored in a refrigerator so that no fading of the latent image would be expected to take place during this period.

(*) Now at the University of Delhi.

TABLE I.

Stack	Exposure date	Minimum (plateau blob density/50 μm)
π_1^- Bristol 4.5 GeV (*) π-meson stack	Jan. 1955	11.0
K_1^+ European Collaboration	March 1955	8.0
K_2^+ » »	June 1955	7.5
K_1^- » »	March 1956	8.0
K_3^+ » »	Dec. 1956	9.0
K_2^- » »	Oct. 1956	8.0
π_2^- Bristol 4.5 GeV π-meson stack	Jan. 1957	7.5

(*) This stack was shipped directly San Francisco-Europe by the Polar route.

In our experience, emulsions exposed in England have yielded values for minimum ionization which are significantly higher than those obtained in Berkeley exposed stacks. Example figures are given in Table II.

TABLE II.

Stack	Exposure date	Minimum (plateau blob density/50 μm)
100 MeV π^+ stack $_1$	Nov. 1956 Liverpool	12.0
100 MeV π^+ stack $_{2,3}$	April 1957 Liverpool	(2) 11.5
		(3) 12.0

The conclusion to which we were drawn was that that the difference was due to the fading of the latent image during the journey Berkeley-Bristol. To try and eliminate this effect we have recently refrigerated a number of stacks after exposure and on their return journey from Berkeley. Also we have chosen an air route from Berkeley which eliminates the trans-shipment in New York (B.O.A.C. direct flight to London). The stacks were packed in an insulated box in the presence of solid carbon di-oxide refrigerant. On arrival in Bristol the dry ice was still present after a journey of only 48 hours from Berkeley to Bristol.

The minimum ionization values obtained in these stacks are shown in Table III.

TABLE III.

Stack	Exposure date	Minimum (plateau blob density/50 μm)
K_4^+ Bristol-Dublin-Padua stack	July 1957	11.5
K_5^+ Bristol-Dublin-Padua test stack	»	11.4
K_3^- Delhi stack	»	11.6

The K_4^+ and K_3^- stacks were from the same pouring at Ilford INC.

It is concluded that refrigeration is certainly necessary for emulsion stacks when they are shipped over large distances by public transport and that although the fading of the latent image during transit may not be the complete explanation of the low minimum values obtained in recent years, it was a contributory factor.

* * *

Our large debt to Dr. C. WALLER and his colleagyes at Ilford INC for preparing all the emulsion stacks is gratefully acknowledged. We would like to thank Prof. E. J. LOFGREN for the Bevatron exposures and Professor J. M. CASSELS for the Liverpool exposures. We are greatly indebted to Drs. D. STORK and J. H. MULVEY for considerable help in recent Bevatron exposures. Thanks are also due to many members of H. H. Wills Physical Laboratory who have helped in the tedious processing work on many occasions and Mr. C. PROCOPIO of the University of Padua for help in the processing of the K_4^+, K_5^+ and K_3^- stacks.

RIASSUNTO (*)

Fondandosi su recenti osservazioni, si suggerisce di mantenere refrigerati dopo l'esposizione i pacchi di emulsioni Ilford G5 se debbono essere spedite a grandi distanze coi pubblici mezzi di trasporto.

(*) *Traduzione a cura della Redazione.*

Mass Selection Rules in the Bilocal Theory - I.

E. Minardi

Società Nazionale Cogne - Aosta
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

(ricevuto il 12 Aprile 1957 (*))

In the bilocal theory of elementary particles previously developed [1] the masses are zeros of the Bessel functions of half-odd order

$$J_{k+\frac{1}{2}}(ml) = 0 \,, \tag{1}$$

where l is the universal length and k is zero or a positive integral number connected with the internal total angular momentum of the particle in such a way that the internal total angular momentum is given by k in the case of bosons and by $k-\frac{1}{2}$ (with $k\neq 0$) in the case of fermions [2].

The masses obtained by conditions (1) for $k\neq 0$ can be either bosons or fermions so that a selection rule is necessary to distinguish between the two kinds of particles.

In the previous work it was assumed that bosons must have an even k while fermions an odd k. However the number of masses predicted by condition (1) and by the above selection rule is greater than the number of observed masses of bosons and fermions. In this letter we suggest a simple selection rule which excludes the majority of the non-observed masses.

In order to define the internal intrinsic parity (i.e. the parity with respect to reflection in $\boldsymbol{\eta}$ space) of an elementary particle having a mass which is obtainable from one of the zeros of equation (1), let us consider all the zeros of (1) for all k in the increasing order of the masses and let r be the position number (1, 2, 3, ...) of the zeros in this succession. Then we define the internal parity of the particles as given by $(-1)^r$.

Now, if $m=\pm(2k+1)/2$, $(k=0, 2, 4, ...)$ is the index of the Bessel functions for bosons (in the case of bosons we shall consider also a Bessel function of negative order), we allow only those bosons for which the relation

$$(-1)^r = (-1)^{m-(k+1)/2} = (-1)^{\pm(2k+1)/2-(k+1)/2} \,,$$

is satisfied.

In the case of fermions we allow only

(*) Il testo di questa lettera è stato rielaborato dall'Autore e licenziato nella forma definitiva attuale nell'Ottobre passato. *N. d. R.*

[1] E. Minardi: *Nuovo Cimento*, **11**, 694 (1954); **12**, 950 (1954); **3**, 968 (1956); **4**, 1127 (1956).

[2] The number k enters in the internal equation through the product $k(k+1)$. In the case of fermions k can be an integral negative; however the same spherical Bessel function $J_{A+\frac{1}{2}}$ is obtained with $k=A$ or $k=-A-1$ $(A>0)$ so that for each odd positive k an even negative k must correspond.

TABLE I. – *Bosons: positive even k and $k=0$.*

Internal total angular momentum	Allowed internal intrinsic parity	k	Internal function	Allowed or excluded masses and corresponding internal parities (the allowed masses are given in boldface type)				
0	—	0	$J_{1/2}$	**265**; $(-1)^1$	795; $(-1)^4$	1325; $(-1)^{18}$	1855; $(-1)^{28}$	...
0	+	0	$J_{1/2}$	530; $(-1)^1$	1060; $(-1)^4$	1590; $(-1)^{10}$	2120; $(-1)^{17}$	...
2	—	2	$J_{5/2}$	**972**; $(-1)^3$	1533; $(-1)^8$	2080; $(-1)^{16}$	**2615**; $(-1)^{27}$	...
4	+	4	$J_{9/2}$	1380; $(-1)^7$	1975; $(-1)^{15}$	2537; $(-1)^{25}$	3085; $(-1)^{37}$	...
6	—	6	$J_{13/2}$	1768; $(-1)^{12}$	**2390**; $(-1)^{23}$	**2975**; $(-1)^{35}$	3530; $(-1)^{46}$	...
...	...	...	...	...	...	...	...	...

TABLE II. – *Fermions: positive odd k (or negative even k).*

Internal total angular momentum	Allowed internal intrinsic parity	k	Internal function	Allowed or excluded masses and corresponding internal parities (the allowed masses are given in boldface type)				
$\frac{1}{2}$	—	1	$J_{3/2}$	758; $(-1)^2$	1302; $(-1)^6$	**1839**; $(-1)^{13}$	2365; $(-1)^{22}$	...
$\frac{5}{2}$	+	3	$J_{7/2}$	1180; $(-1)^5$	1757; $(-1)^{11}$	**2310**; $(-1)^{20}$	**2860**; $(-1)^{32}$	...
$\frac{9}{2}$	—	5	$J_{11/2}$	**1578**; $(-1)^9$	**2185**; $(-1)^{19}$	2758; $(-1)^{30}$	3310; $(-1)^{44}$	...
$\frac{13}{2}$	+	7	$J_{15/2}$	**1965**; $(-1)^{14}$	**2600**; $(-1)^{26}$	3170; $(-1)^{41}$	3760; $(-1)^{53}$	...
...	...	...	...	...	...	...	...	...

those particles for which the relation $(-1)^r = (-1)^{2m-(k-1)/2} = (-1)^{\frac{3}{2}(k+1)}$ holds, where $m=(2(k-1)+3)/2$, $(k=1, 2, 5,...)$ is the index of the Bessel functions for fermions.

By passing from one order of the Bessel function to the next higher order the internal parity allowed by these selection rules becomes even or odd alternatively.

In Tables I and II the lowest masses predicted by condition (1) with a $2.30 \cdot 10^{-13}$ cm value for the universal length, are given.

The allowed masses are given in boldface type.

All Bessel functions of negative order, not included in the Tables I and II (e.g. all but $J_{-\frac{1}{2}}$), have a singularity at $r=0$ which gives infinite mass in every volume containing the origin. We allow only spin 0 particles with higher mass quantity in this volume and thus we exclude the $J_{\frac{1}{2}}$ internal function (3).

It is noted that the calculated masses, other than the neutron, are slightly higher than the observed values. This may suggest that the universal length is slightly higher than the $2.3 \cdot 10^{-13}$ cm value. The difference between the neutron mass obtained with the higher value of l and the observed mass may be due to self-energy corrections.

To the allowed masses of Table II we must add a vanishing mass with spin $\frac{1}{2}$ and a purely electromagnetic mass $m = 3e^2/5l$ with the same spin, as was shown in a previous work (1). These masses are not obtained with condition (1), but with a solution of the spinorial internal equation whose matrix elements are *all* discontinuous at $r = 1$. We believe that this kind of solutions is characteristic of leptons. Moreover it appeared in previous work that a characteristic difference between bosons and fermions is that at least some matrix element of the internal spinorial function can be discontinuous at $r=1$, while the internal function of bosons is always continuous. Thus a vanishing mass with integral spin and with internal field is excluded by the requirement of continuity of the internal function of bosons.

A photon (with zero mass) may be considered in our bilocal theory as a particle without internal field.

The purely electromagnetic mass must obviously be identified with the electron, but the calculated value is only $\frac{3}{4}$ of the measured mass. This discrepancy may be due to the approximations made in calculating the mass or to some other reasons (4).

This theory is now faced with the explanation of the reasons for which some allowed masses are not observed. Moreover the existence of the muon is not predicted by our spectrum, a problem which is perhaps connected with the introduction of a new interaction.

It is an obvious observation that many problems now arise for testing and developing the present bilocal theory. These problems are mainly concerned with the connections between the bilocal theory of the elementary particles and their interactions.

It is then concluded that, if it seems rather unlike that the above numerical results are purely accidental, much further theoretical and experimental work is still to be done in order to establish in a conclusive manner the theory outlined by the autor in the last few years.

* * *

Thanks are due to Prof. G. WATAGHIN for useful suggestions.

(3) The internal intrinsic parity of the particles corresponding to the Bessel function $J_{-\frac{1}{2}}$ is defined as above, by ordering, in an increasing succession, all zeros of the Bessel functions of negative half-odd order.

(4) For example, a self-energy correction due to the interaction with an other field.

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Note on the Polarization of the Muon in $K_{\mu 3}$.

S. Furuichi, S. Sawada and M. Yonezawa
Department of Physics, Hiroshima University - Hiroshima

(ricevuto il 20 Novembre 1957)

Attempting to clarify the type of decay interaction in $K_{\mu 3}$ ($K^+ \to \mu^+ + \bar{\nu} + \pi^0$), the theoretical analyses of the energy spectra and longitudinal polarization of the muon have been presented [1-4,5]. Although the experimental investigation of these quantities will serve for the determination of the type of weak interaction, these analyses contain some assumptions. They are:

(I) muon and neutrino are created at one-point;

(II) the effective coupling constants which appear in the transition matrix can be approximated by momentum independent constants.

The assumption (I) may be acceptable, since the radiative correction which violates (I) is of electromagnetic nature and brings only an error of order $e^2/\hbar c \sim \sim 1/137$ [6].

However we can not infer to the validity of the assumption (II). The assumption has not clear physical ground. It may be true and may not be true. The judgement on the validity will be postponed until the details of the nature of strong interaction involved become plain enough.

To avoid this defect it is proposed to examine the longitudinal polarization of the muon with fixed μ-ν angle and muon energy [7,8].

An experiment with a heavy-liquid bubble chamber that can detect neutral mesons as well as muons is in progress [9] and the analysis will produce fruitful results in future.

(1) S. Furuichi, T. Kodama, S. Ogawa, Y. Sugahara, A. Wakasa and M Yonezawa: *Prog. Theor. Phys*, **17**, 89 (1957).

(2) S. Furuichi: *Nuovo Cimento*, in press.

(3) J. Werle: *Nuclear Phys.*, **4**, 171 (1957).

(4) S. Furuichi, S. Sawada and M. Yonezawa: *Nuovo Cimento*, in press.

(5) These analyses have been performed under the view-point that weak Fermi interaction is the primary weak interaction participating in the decay. We mean the direct interaction among four fermions (no Konopinski-Uhlenbeck type) by the term « Fermi interaction ». The possibility of K-U type interaction has been discussed by Y. Miyachi; *Prog. Theor. Phys.*, **19**, 112 (1958).

(6) The discussion of radiative correction will be given elsewhere in near future.

(7) Ref. (4) and J. Werle: preprint. The authors thank Dr. J. Werle for his kind sending the preprint before its publication.

(8) In this case the assumptions (I) and (II) and also the Fermi interaction hypothesis can be removed if we permit a little complication.

(9) Michigan group.

Still there may exist some difficulties: *a*) it requires a great number of events to give a conclusion, *b*) the dependency of polarization on energy and angle is very sensitive for some type of interaction.

In this note we wish to show that the second assumption (II) can be removed without introducing the above two difficulties *a*) and *b*), if we measure the polarization of the muon along the pion propagation direction for a fixed pion momentum. The analysis of polarization along this direction will give us some definite information about the decay interaction. The polarization along the pion direction is given by

$$P(l_0, \boldsymbol{l}/l) = \frac{a(l_0, \boldsymbol{l}/l) - a(l_0, -\boldsymbol{l}/l)}{a(l_0, \boldsymbol{l}/l) + a(l_0, -\boldsymbol{l}/l)},$$

where notations keep correspondence with those of our previous article ([10]). In order to calculate $P(l_0, \boldsymbol{l}/l)$, the assumption (II) is no longer required. This is because the argument of the effective coupling constant is $k_\mu l_\mu$, i.e. $m_\varkappa l_0$ and has no angular dependency. It is noted that the denominator of the expression of polarization $a(l_0, \boldsymbol{l}/l) + a(l_0, -\boldsymbol{l}/l)$ is identified with the neutral pion distribution.

Since the general expression of polarization is rather lengthy, we give only the curves of polarization for pure interaction when the neutrino is of two components ([11]) and the spin of the K-meson is 0 in Fig. 1.

There are several features in the polarization of the muon along the pion direction that makes the comparison between theory and experiment easy and promising.

([10]) Ref. ([4]).

([11]) The present experimental evidence does not seem to favor for the simple two component theory of neutrino. However the violation of space parity conservation seems to show maximum asymmetry. The information obtained about the two-component theory is applicable to other cases such as the twin neutrino theory.

A) For pure interaction (and $F = \pm F'$) the polarization is independent of the form of F ([12]).

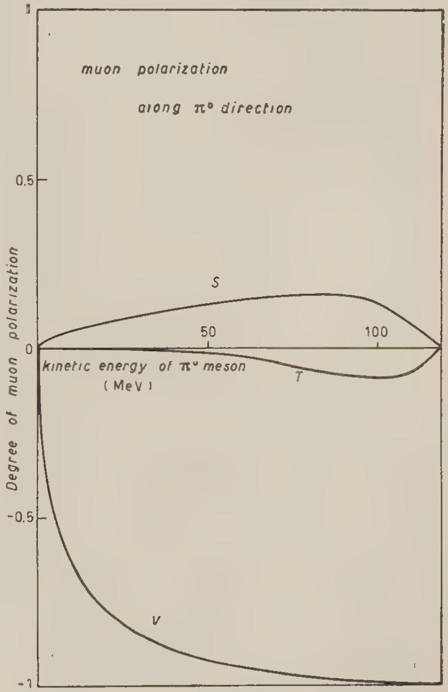

Fig. 1.

B) There is a large difference in the magnitude of polarization among *S*, *V* and *T*. The difference is larger than the one in the case of the longitudinal polarization.

C) The fact that the dependency of energy is weak and rather simple may be used to deduce a rough information from integrated (over energy) data without introducing large errors.

It is here emphasized that even in the case of mixed interaction, our ignorance of the structure of the effective coupling constant can be somewhat circumvented by adjusting them with the energy distribution of pion.

The detailed accounts of the present analysis will be published elsewhere.

([12]) F' is the effective coupling constant of parity non-conserving interaction.

Solution of Boltzmann Equation for Multiple Scattering of Electrons.

G. Molière (*)

CERN - Geneva

(ricevuto il 27 Novembre 1957)

At present, many investigators studying the polarization of β-rays use the Mott or the Møller scattering in thin foils as detectors. A serious cause of error thereby arises from the plural or multiple scattering of the electrons inside the foils. An exact determination of this influence is therefore highly desirable.

Subsequent to a previous investigation by W. Bothe (1), Bethe, Rose and Smith (2) have treated the problem of a beam of electrons perpendicularly incident on a plane-parallel plate, passing through it and partly being scattered back, under the influence of pure multiple scattering. Their treatment was based on the approximation in which the Boltzmann integro-differential equation is replaced by a diffusion-differential equation. The first step toward a more general treatment was done by M. C. Wang and E. Guth (3). Further progress was achieved by R. Mertens (4). Re-investigating this problem we arrived at a solution on the following lines (5).

The motion of electrons inside the plate is governed by the Boltzmann equation which, in our case of axial symmetry and under stationary conditions, may be written in the simple form (+):

$$\frac{\mu\, \partial I(\mu, x)}{\partial x} = - G(\mu) * I(\mu, x) \,, \tag{1}$$

where the abbreviation is used:

$$G(\mu) \equiv N[\sigma_0 \delta(\mu - 1) - \sigma(\mu)] \,, \tag{1'}$$

(*) The work reported here was already started at the Faculdade de Filosofia, Ciências e Letras, São Paulo.

(+) The case of oblique incidence may also be included in our treatment. We have to admit in this case that the intensity $I(\mu, \alpha, x)$ in (1) depends also on the azimuth α.

(1) W. Bothe: *Zeits. f. Phys.*, **54**, 161 (1929).

(2) H. A. Bethe, M. E. Rose and L. P. Smith: *Amer. Phil. Soc.*, **78**, 573 (1938), quoted as BRS.

(3) M. C. Wang and E. Guth: *Phys. Rev.*, **84**, 1092 (1951).

(4) R. Mertens: *Wis- en Natuurkundig Tijdschrift, Ghent*, 30e *Jaargang, Supplement.*

(5) G. Molière: to be published in *Zeits. f. Naturfor.*

and where by the asterisk is denoted the « folding product at the unit sphere » as defined by (2) below. Further, in our notation, $I(\mu, x)$ is the intensity of electrons, x the coordinate perpendicular to the surfaces of the plate, μ the cosine of the angle of deflection of the electrons from the positive x-direction, N the density of scattering atoms in the plate, δ the Dirac function and, respectively, σ_0 and $\sigma(\mu)/2\pi$ the total and the differential scattering cross-section.

The type of folding product occurring in (1) is defined for two arbitrary functions $g(\mu)$ and $f(\mu)$ by the integral:

$$g(\mu)*f(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha' \int_{-1}^{+1} \mathrm{d}\mu' g(\mu'')f(\mu') , \tag{2}$$

with

$$\mu'' = \mu\mu' + [(1 - \mu^2)(1 - \mu'^2)]^{\frac{1}{2}} \cos(\alpha - \alpha') . \tag{2'}$$

For folding products of this type the special folding theorem holds:

$$(g*f)_l = g_l f_l , \tag{3}$$

connecting with each other the Legendre coefficients $g_l \ (= \int_{-1}^{+1} g(\mu)P_l(\mu)\,\mathrm{d}\mu)$, f_l and $(g*f)_l$ of the corresponding functions of μ.

Introducing in the left hand side in (1) the « current function » $J(\mu, x) = \mu I(\mu, x)$ and applying (3) we immediately get the infinite set of linear equations (*):

$$\frac{\mathrm{d}J_l(x)}{\mathrm{d}x} = - G_l I_l(x) . \tag{4}$$

Here, the Legendre coefficients G_l of $G(\mu)$ given by (1′) characterising the single scattering law are identical to those introduced already by GOUDSMIT and SAUNDERSON ([6]). The two sets of Legendre coefficients $I_l(x)$ and $J_l(x)$ ($l = 0, 1, 2, 3, \ldots$) are related to each other by the equations:

$$J_l(x) = [lI_{l-1}(x) + (l+1)I_{l+1}(x)]/(2l+1) . \tag{5}$$

In the case of small angle scattering (where only $\mu \approx 1$ contributes) (5) may be replaced by the approximate equation $I_l(x) \approx J_l(x)$ which, introduced into (4), leads to the set of equations obtained for this case by GOUDSMIT and SAUNDERSON ([6]).

In the general case, the exact relation (5) has to be used to remove from (4)

(*) If the intensity depends on the azimuth α (see the preceding foot note) we have to include associated Legendre coefficients $I_{l,m}(x) \left[= (1/2\pi)\int_0^{2\pi} \mathrm{d}\alpha \int_{-1}^{+1} \mathrm{d}\mu I(\mu, \alpha, x) P_l^m(\mu) \cos m\alpha\right]$ and $J_{l,m}(x)$. For such ones, as a generalization of (3), the folding theorem $(g * f)_{l,m} = g_l f_{l,m}$ holds, which applied to (1) leads to the set of equations $\mathrm{d}J_{l,m}/\mathrm{d}x = - G_l I_{l,m}$. These form separate partial sets for each value of m, including (4) for $m = 0$. Their solutions for $m \neq 0$ are all of the « exponential » type: $I(\mu, \alpha, x) = C \exp[\mp k_n^{(m)} x]\varphi_n^{(m)}(\pm \mu) \cos m\alpha$, where the $\varphi_n^{(m)}(\mu)$ are « associated » functions corresponding to the $\varphi_n(\mu)$ considered below.

([6]) S. A. GOUDSMIT and J. L. SAUNDERSON: *Phys. Rev.*, **57**, 24, 552, **58**, 36 (1940).

half the number of coefficients $I_l(x)$, $J_l(x)$ ($l = 0, 1, 2, 3, ...$). The most appropriate way of doing this consists in the elimination from (4) of all the even coefficients $I_{2\nu}(x)$ and $J_{2\nu}(x)$ (viz. their derivatives). This procedure corresponds to the special representation of the intensity

$$(6) \qquad I(\mu, x) = \sum_{\nu=0}^{\infty} (2\nu + \tfrac{3}{2}) \left[I_{2\nu+1}(x) + \frac{J_{2\nu+1}(x)}{\mu} \right] P_{2\nu+1}(\mu) ,$$

where only odd coefficients I and J are used. It is suggested by the following considerations.

From the property of the folding operation in (1) of being self-adjoint

$$\left(\text{i.e.} \int_{-1}^{+1} \varphi(\mu)[G(\mu) * \psi(\mu)] \, \mathrm{d}\mu = \int_{-1}^{+1} \psi(\mu)[G(\mu) * \varphi(\mu)] \, \mathrm{d}\mu \right),$$

together with the occurrence of the factor μ in the left hand side of (1), it follows that the linearly independent solutions of (1), with respect to the variable μ, are essentially « orthogonal with weight μ ». These solutions, as functions of μ and subject to the transformations leaving the « inner product with weight μ » invariant, form a certain (non-Hilbert) function space in which the odd parts of its elements and μ times their even parts, respectively, behave as co- and controvariant vectors. This fact suggests starting from a special representation in which both, the odd parts of the intensity I and of the « current » $J = \mu I$ are developed with respect to the same orthogonal system for odd functions. Choosing the odd Legendre polynomials, one is led to (6).

The system of linear differential equations for the $I_{2\nu+1}$ and $J_{2\nu+1}$ obtained from (4) with the help of (5) splits into two independent partial systems, one being formed by I_1 and J_1 and the other by the remaining odd coefficients I and J. The former leads to exactly the same singular solution, linear in μ and x, which was obtained by BRS in the diffusion case. (The occurrence of this singular solution is a consequence of $G_0 = 0$ and is connected wth the conservation of the number of electrons). The latter partial system leads to solutions which are also of the same type as in the diffusion case: $I(\mu, x) = \text{const}\, \varphi_n(\pm\mu) \exp[\mp k_n x]$. The eigenvalues k_n and the eigenfunctions $\varphi_n(\mu)$ are determined by an infinite secular equation which contains only three non-vanishing diagonal lines, converges rapidly and may be solved by a simple numerical procedure.

As a numerical example, we have chosen the case of the diffusion differential equation which is included in our general treatment by the special (simplified) choice, $G_l = l(l+1)/\lambda$ (*), of the Goudsmit-Saunderson coefficients. On this basis the first five eigenvalues k_n (+) and the corresponding eigenfunctions $\varphi_n(\mu)$ have been computed. For comparison with the results thus obtained a higher order asymptotic treatment was applied to the diffusion differential equation which was carried out in detail by H. Joos and P. Leal Ferreira (see the succeeding Letter to the Editor, where this method and its main results are described separately).

(*) λ is the « transport mean free path » introduced by BRS. It is interesting to note that λ coincides exactly with $2/G_1$ where G_1 is the first Legendre coefficient of the function given by (1').

(+) See column II of the table in the succeding letter. [*Nuovo Cimento*: 7, 724 (1958)].

The fulfilment of the boundary conditions at the surfaces of the plate has been carried out in terms of rapidly convergent series containing products of exponentials $\exp[-k_n d]$ where d is the thickness of the plate. If all these exponentials are put to zero, our solution corresponds to the asymptotic one given by BRS. Their results in the diffusion case, by our computation, are confirmed in this asymptotic region and extended to smaller d where the transmission coefficient is found to tend rapidly toward the value one (see the figure).

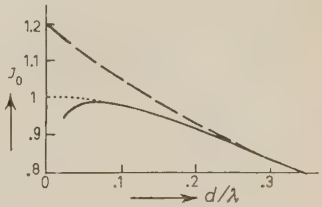

Fig. 1. – The transmission coefficient in diffusion approximation. – – – asymptotic, all exponential terms neglected. — the first power of $\exp[-k_1 d]$ only taken into account. ... qualitative extrapolation to small d.

The diffusion approximation is clearly seen to be adequate, just in the asymptotic domain of large thickness to which it was applied by BRS while, for smaller d, sensible changes are to be expected from the application of the correct G_l. For larger thickness, of course, the asymptotic behaviour will be decisively affected by the energy loss due to ionization which, strictly speaking, is never negligible, except for very small thickness. Its influence, however, will remain comparatively small within a range of thickness including an appreciable part of the above « asymptotic » region, provided the initial (kinetic) energy of the electrons does not exceed some mc^2, say. Compare L. V. SPENCER (7) who recently has suceeded in treating the motion of electrons in an infinite medium under the combined influence of multiple scattering and energy loss.

(7) L. V. SPENCER: *Phys. Rev.*, **98**, 1597 (1955).

IL NUOVO CIMENTO Vol. VII, N. 5 1° Marzo 1958

Generalized WKB-Method of Asymptotic Expansion.

G. Molière (*)

CERN - Geneva

(ricevuto il 27 Novembre 1957)

As described in the preceding Letter to the Editor, the theory of electrons passing through a plate under the influence of multiple scattering through large angles was worked out in general form. The numerical evaluation was confined to the simplified case equivalent to replacing the Boltzmann equation by the diffusion differential equation (+):

$$\mu \frac{\partial I(\mu, x)}{\partial x} = \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} I(\mu, x)\,, \tag{1}$$

which, for the exponential solutions $I(\mu, x) = \text{const}\, \varphi_n(\mu) \exp[-k_n x]$ leads to the eigenvalue equation (×):

$$\left[\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} + k_n \mu\right]\varphi_n(\mu) = 0\,. \tag{2}$$

For comparison with the results obtained by this rigorous evaluation, it appeared worth while to apply to (2) a higher order asymptotic treatment, extending the procedure of Bethe, Rose and Smith (1).

For the eigenvalues k_n, in this way, the asymptotic formula was obtained (as an improvement of the one given by BRS):

$$k_n \approx A\left[\left(n+\frac{1}{2}\right)^2 - \frac{5}{12\pi} - \frac{C}{(n+\frac{1}{2})^2}\right], \tag{3}$$

(*) The work reported here was already started at the Faculdade de Filosofia, Ciências e Letras, São Paulo.

(+) The transport mean free path λ is used here as unit of length x.

(×) The « associated » functions $\varphi_n^{(m)}$ obey an equation which arises from (2) by adding a term $-m^2/(1-\mu^2)$ in the square bracket. For the corresponding eigenvalues $k_n^{(m)}$ we have also derived an asymptotic formula of the form (3), including (3) as a special case.

(1) H. A. Bethe, M. E. Rose and L. P. Smith: *Proc. Amer. Phil. Soc.*, **78**, 573 (1938), quoted as BRS.

with $A = 6.875186$ and $C = 0.00825$. As the following table shows, the agreement of (3) with the exact numerical figures is excellent, the relative error even for $n = 1$ and 2 being as small as $3 \cdot 10^{-4}$ and 10^{-5}, respectively. Column II of our table gives the exact numerical values k_n obtained by the methods described in the preceding article. Column III containes the figures of column II devided by the constant A of our asymptotic formula (3) while the figures of column IV correspond to the square bracket in (3).

I	II	III	IV
n	k_n, exact	k_n/A, exact	k_n/A, asymptotic
1	14.528004 (*)	2.11311	2.11370
2	42.04855	6.11599	6.11605
3	83.30444	12.11669	12.11670
4	138.3078	20.11695	20.11696
5	207.0606	30.11709	30.11710

(*) The first eigenvalue k_1 had already been determined by BRS using direct numerical integration of equation (2). Their result, 14.476, is slightly too low compared with ours.

H. JOOS and P. LEAL FERREIRA [2] have elaborated the asymptotic method in detail and have also applied it to derive asymptotic expressions for the eigenfunctions $\varphi_n(\mu)$, which were also found to agree closely with the rigorous results. The main features of this useful method may be shortly described here, as follows.

Suppose we wish to express a function $f(\mu)$ asymptotically in terms of another, well-known, function $F(\eta)$. Let these functions fulfil differential equations, both linear homogeneous of the second order in normal form:

$$f''(\mu) + [kA(\mu) + B(\mu)]f(\mu) = 0\,, \tag{4}$$

and

$$F''(\eta) + C(\eta)F(\eta) = 0\,, \tag{5}$$

the former containing the large parameter k. (In our case we have $f(\mu) = \sqrt{1-\mu^2}\,\varphi_n(\mu)$ for which from (2) follows a normal form equation). To express $f(\mu)$ asymptotically in terms of $F(\eta)$, introduce into (4) the following relation between η and μ:

$$f(\mu) = [\eta'(\mu)]^{-\frac{1}{2}}F[\eta(\mu)]\,, \tag{6}$$

and eliminate F'' from the result by means of (5). $F(\eta)$, then, occurs as a common factor of all terms and may be dropped which leads to the differential equation for $\eta(\mu)$:

$$\eta'^2 C(\eta) - kA(\mu) + \text{small terms} = 0\,, \tag{7}$$

[with small terms $= B(\mu) + \frac{3}{4}(\eta''/\eta')^2 - \frac{1}{2}(\eta'''/\eta')$].

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[2] H. JOOS and P. LEAL FERREIRA, to be published in *Anais Acad. Brasil. Cien.*, Rio de Janeiro.

In most practical cases, $C(\eta)$ will be of a particularly simple form. Sometimes, the first term in (7) can be simplified by splitting off from it a small term which is combined with the rest. In suitable cases, then, the solution of (7) can be given in successive steps of approximation, neglecting in the first step all the small terms and expressing them in the higher steps by the result of the foregoing. The function $F(\eta)$ has to be chosen in different ways to get asymptotic representations valid in different regions of the variable μ. For the exponential or trigonometric function the method reduces to the WKB-method. To get a representation valid aroud the « umkehr » point, $\mu = 0$, one has to choose $F(\eta) = \eta^{\frac{1}{2}} Z_{\frac{1}{3}}[\frac{2}{3}\eta^{\frac{3}{2}}]$ where $Z_{\frac{1}{3}}$ is a certain Bessel function of the order $\frac{1}{3}$. The method in this case is equivalent to that of LANGER (3). In the neighbourhood of the end points, $\mu = \pm 1$, finally, a Bessel function of the order zero has to be chosen. It can be shown that the different representations, obtained in this way, are asymptotically equivalent.

As an illustration, we may consider the choice $f(\theta) = (\sin\theta)^{\frac{1}{2}} P_l(\cos\theta)$ (with θ replacing the above μ) and $F(\eta) = \eta^{\frac{1}{2}} J_0(\eta)$ which gives

$$\eta(\theta) = (l + \tfrac{1}{2})\,\theta + (\theta^{-1} - \cot\theta)/(8l + 4) + \dots . \tag{8}$$

Introduced into (6), this leads to the following asymptotic representation for the Legendre polynomials:

$$P_l(\cos\theta) = \left[\frac{\theta}{\sin\theta}\right]^{\frac{1}{2}} \left[1 - \frac{\alpha(\theta)}{(2l+1)^2}\right] J_0[\eta(\theta)]\,, \tag{9}$$

where $\eta(\theta)$ is given by (8) and

$$\alpha(\theta) = 1 - (\theta^{-1} - \cot\theta)(2\theta^{-1} + \cot\theta) \simeq 2\theta^2/45 + \dots . \tag{10}$$

(The term containing $\alpha(\theta)$ gives in most cases only a slight correction). (9) is an improvement of an asymptotic formula for $P_l(\cos\theta)$ which had been formerly indicated by the author (4) without knowledge, at that time, that it was originally due to HILB (5).

(3) R. E. LANGER: *Phys. Rev.*, **51**, 669 (1937); for more recent literature see: F. W. J. OLVER: *Phil. Trans. Roy. Soc.*, London (A), **247**, 307 (1954).

(4) G. MOLIÈRE: *Zeits. f. Naturfor.*, **2a**, 133 (1947), eq. (A.1).

(5) E. HILB: *Math. Zeits.*, **5**, 17 (1919); **8**, 79 (1920); see also: G. SZEGÖ: *Orthogonal Polynomials*, in *Amer. Math. Soc. Coll. Publ.*, vol. **23**, formula (8.21.7).

Proposed Experiment Bearing Directly on Helicity of Neutrinos (*).

L. A. Page (+)

Institute of Physics, University of Uppsala - Uppsala

(ricevuto il 12 Dicembre 1957)

At the time of the first demonstration of parity non-conservation in β-decay [1,2] and up until now [3], the sign of the helicity of neutrinos has always been inferred from *two* measured signs: (i) the sign of $\langle \boldsymbol{p}_\nu \cdot \boldsymbol{K} \rangle$, and (ii) the sign of $\langle \boldsymbol{j} \cdot \boldsymbol{K} \rangle$. The neutrino momentum is $\boldsymbol{p}_\nu$, the special direction $\boldsymbol{K}$ is usually the direction of emission of the associated β-ray, and $\boldsymbol{j}$ denotes whatever angular momentum is measured: the angular momentum of a nucleus, the spin of a β-particle, or the «angular momentum» of a circularly-polarized photon (be it γ-ray, bremsstrahlung photon or annihilation quantum). Measurements (i) and (ii) have customarily been made separately: by different observers in different laboratories, but more importantly—on different decays, and often using β-particles of opposite charge. The purpose of this note is to suggest an experimental approach whereby, in effect, $\boldsymbol{K}$ would be replaced by $\boldsymbol{p}_\nu$. Thus one would measure $\langle \boldsymbol{j} \cdot \boldsymbol{p}_\nu \rangle$ which could then be interpreted *directly* in terms of neutrino helicity without using an additional sign from a different experiment.

In an allowed electron capture by an unpolarized nucleus leading via the G-T interaction into a (excited) state of non-zero angular momentum, it is clear from conservation of angular momentum that, should the emitted neutrino be polarized [1], the final nucleus will in general be polarized with respect to $\boldsymbol{p}_\nu$. Considering the circular polarization [4] of a

(*) Work supported in part by Office of Ordnance Research, U. S. Army.

(+) Visiting Guggenheim Fellow, Permanent address: Physics Department, University of Pittsburgh, Pittsburgh, Pennsylvania.

(1) T. D. Lee and C. N. Yang: *Phys. Rev.*, **105**, 1671 (1957).

(2) C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes and R. P. Hudson: *Phys. Rev.*, **105**, 1413 (1957); R. L. Garwin, L. M. Lederman and M. Weinrich: *Phys. Rev.*, **105**, 1415 (1957).

(3) See the several parity reviews presented at the International Conference on Nuclear Structure, Rehovoth, September 1957, (to be published).

(4) See J. C. Wheatley, W. J. Huiskamp, A. N. Diddens, M. J. Steenland and H. A. Tolhoek: *Physica*, **21**, 841 (1955), regarding directional correlation between nuclear polarization and circularly-polarized gammas. Or, considering the overall directional correlation of interest here, namely that between *neutrino* and polarized γ-ray, we note that it rests on those physical principles responsible for the better known directional correlations which exist between β-ray and polarized γ-ray; see F. Boehm and A. Wapstra: *Phys. Rev.*, **106**, 1364 (1957); and K. Alder, B. Stech and A. Winther: *Phys. Rev.*, **107**, 728 (1957).

subsequently emitted γ-ray, one notes that such polarization goes linearly ([5]) with photon energy in the laboratory, being zero at the center of the energy distribution. Since the total polarization of such a γ-ray is zero ([6]), what is needed is an analyzer for circular polarization which is sensitive to photons of average energy different from the average energy of the emitted photons. It is here suggested that the nuclear resonance fluorescence process might be used. There are authenticated cases ([7]) where a ground-state γ-ray emitted by a nucleus after electron capture can be in resonance with the same nuclide « at rest » in the laboratory, by virtue of the neutrino recoil. The energetic requirement is $p_\nu/k > 1$. It remains then to analyze for circular polarization *either* those gammas to be resonance-scattered *or* the already resonance-scattered photons. In the former case a transmission-type analyzer would be required so as to leave undisturbed the photon energy. In the latter case the angle of the resonance scattering would have to be so chosen as to preserve ([8]) most of the circular polarization, in magnitude at least.

The essence of the above considerations is that the sign of helicity of the neutrino emitted in the electron-capture process may be inferred from a measurement on the sign of helicity of the « resonance » γ-rays. This is independent of whether the tensor interaction or the axial-vector interaction prevails. One should know, however, the j-values of the three successive states involved, for a correct interpretation of sign.

([5]) This is because (in magnitude) the Doppler-shift and the photon polarization are both proportional to $\hat{\boldsymbol{p}}_\nu \cdot \hat{\boldsymbol{k}}$, where $\boldsymbol{k}$ is the photon momentum. The maximum polarization will depend on a factor taking into account the j-values of the states involved. The slope, polarization versus energy, depends further on k, p_ν and the mass of the nucleus.

([6]) In contrast to the inner-bremsstrahlen for electron-capture, R. E. CUTKOSKY: *Phys. Rev.*, **107**, 330 (1957), which may be highly polarized — but as in the other experiments [ref. ([3])] this would not measure directly neutrino helicity.

([7]) F. METZGER: *Phys. Rev.*, **98**, 200 (1955). In the suggested experiment, however, one asks that the recoiling nucleus remember the *direction* of recoil imparted by the neutrino.

([8]) D. R. HAMILTON: *Phys. Rev.*, **74**, 782 (1948).

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo fascicolo è stato licenziato dai torchi il 6-III-1958